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BULLETIN OF THE

AMERICAN MATHEMATICAL SOCIETY

SOME INSTRUCTIVE EXAMPLES IN THE CALCULUS OF VARIATIONS.

BY PROFESSOR OSKAR BOLZA.

(Read before the American Mathematical Society, September 2, 1902.)

In the following note I propose to give some examples which illustrate in a simple manner several points of fundamental importance in the calculus of variations.

§1. *The General Problem and its Assumptions.*

We consider the problem * to minimize the integral

$$I = \int_{x_0}^{x_1} F(x, y, y')dx \tag{1}$$

under the following *assumptions:*

1. The function $F(x, y, z)$ considered as a function of the three independent variables x, y, z is real and regular † in the vicinity of every finite real point $x = a$, $y = b$, $z = c$ for which $x = a$, $y = b$ lies in a given region R of the xy-plane.

2. The functions $y = f(x)$ admitted to consideration satisfy the following conditions:

(a) For the given end values $x = x_0$, $x = x_1$, y takes the given values $y = y_0$ and $y = y_1$ respectively, *i. e.*, the "curves" $y = f(x)$ pass through two given points A: (x_0, y_0) and B: (x_1, y_1);

* Compare for the formulation of the problem Osgood's article "Sufficient conditions in the calculus of variations," *Annals of Math.*, 2nd ser., vol. 2 (1901), p. 105. We deviate, however, in several points from Osgood's assumptions.

† *i. e.* developable into an ordinary power series in $x - a$, $y - b$, $z - c$.

(*b*) y is continuous on the whole interval (x_0x_1), and the first derivative y' exists and is continuous on (x_0x_1) with the possible exception of a finite number of points; in these exceptional points the progressive and regressive derivatives exist and are finite;

(*c*) The "curves" $y = f(x)$ lie, for $x_0 \leqq x \leqq x_1$ in the interior of the region R.

The totality of functions (curves) satisfying these conditions will be denoted by $\mathfrak{M}$.

A function $y = f(x)$ of $\mathfrak{M}$ is said to *minimize* the integral I if there exists a positive quantity ϵ such that the total variation

$$\Delta I = \int_{x_0}^{x_1} F(x, \bar{y}, \bar{y}')dx - \int_{x_0}^{x_1} F(x, y, y')dx$$

is positive for every function $\bar{y} = \bar{f}(x)$ of $\mathfrak{M}$, different from y, for which

$$|\Delta y| < \epsilon,$$

where

$$\Delta y = \bar{y} - y.$$

It is well known that the following conditions are *necessary* for a minimum:

1. The minimizing function $y = f(x)$ must be an *extremal*, *i. e.*, satisfy *Euler's differential equation:* *

$$F_y - \frac{d}{dx} F_{y'} = 0, \tag{I}$$

literal subscripts denoting partial derivatives.

2. *Legendre's condition:*

$$F_{y'y'}\big(x, f(x), f'(x)\big) \geqq 0 \text{ on } (x_0x_1). \tag{II}$$

3. *Jacobi's condition:* †

$$x_1 \leqq x_0' \tag{III}$$

where x_0' denotes the "conjugate" of x_0, x_0 being supposed less than x_1.

---

* See for instance Whittemore, *Annals of Math.*, 2nd ser., vol. 2 (1901), p. 130.

† See Erdmann, *Schlömilch's Zeitschrift*, vol. 23 (1878), p. 367.

4. *Weierstrass's condition:* *

$$\frac{E\big(x, f(x), f'(x), p\big)}{\big(p - f'(x)\big)^2} \geqq 0 \tag{IV}$$

on $(x_0 x_1)$ and for every finite value of p, the function

$$E(x, y, y', p)$$

being defined by

$$\begin{aligned} E(x, y, y', p) &= F(x, y, p) - F(x, y, y') \\ &\quad - (p - y')\, F_{y'}(x, y, y'). \end{aligned} \tag{2}$$

Since by Taylor's theorem

$$E(x, y, y', p) = \frac{(p - y')^2}{2} F_{y'y'}\big(x, y, y' + \theta(p - y')\big) \tag{3}$$

where $0 < \theta < 1$, it follows that (IV) is always satisfied whenever

$$F_{y'y'}\big(x, f(x), p\big) \geqq 0 \tag{II_a}$$

on $(x_0 x_1)$ for every finite value of p.

By (II′), (II_a'), (III′), (IV′), we denote the inequalites (II), (II_a), (III), (IV) after the suppression of the equality sign.

§ 2. *Example Illustrating the Necessity of Weierstrass's Condition.*

After Jacobi had discovered condition (III), it was generally believed that conditions (I), (II′), (III′) were also *sufficient* for a minimum, until Weierstrass proved in 1879 their insufficiency and added his fourth necessary condition.

The following example illustrates this point:

Example I: To minimize the integral

$$I = \int_{x_0}^{x_1} y'^2 (y' + 1)^2 dx. \tag{4}$$

* Weierstrass, Lectures, 1879; compare also Osgood, *l. c.*, p. 118, and Hilbert, *Archiv für Mathematik* (3), vol. 1 (1901), p. 231. The expansion of E according to powers of $p - y'$ shows that the quotient $E(x, y, y', p) / (p - y')^2$ remains regular even when $p = y'$.

Here the extremals are straight lines; let

$$\mathfrak{C}: \qquad y = mx + n$$

be the extremal through the two given points $A: (x_0, y_0)$ and $B: (x_1, y_1)$.

Further, we have

$$F_{y'y'} = 2(6y'^2 + 6y' + 1).$$

Hence

$$F_{y'y'}\big(x, f(x), f'(x)\big) = 2(6m^2 + 6m + 1).$$

The roots of the equation

$$6m^2 + 6m + 1 = 0$$

are

$$m_1 = \tfrac{1}{2}(-1 + 1/\sqrt{3}) = -0.2113\cdots$$

$$m_2 = \tfrac{1}{2}(-1 - 1/\sqrt{3}) = -0.7887\cdots$$

Hence, if we suppose the two given points A and B so situated that

$$\text{either} \quad m > m_1 \quad \text{or} \quad m < m_2,$$

condition (II′) will be satisfied.

Further, condition (III′) is always satisfied; for since the extremals through the point A are straight lines, it follows that there exists no conjugate point on $\mathfrak{C}$, however far we may go.

Finally

$$\frac{E\big(x, f(x), f'(x), p\big)}{\big(p - f'(x)\big)^2} = p^2 + 2(m+1)p + 3m^2 + 4m + 1;$$

it is positive for every p when $m(m+1) > 0$; it can change sign when $m(m+1) < 0$.

Accordingly we have to distinguish the following three cases:

Case I: $-1 < m < 0$.

In this case *the line AB does not minimize the integral I.* This can be seen at once as follows: However small we may select a positive quantity ϵ, we can always draw in the "neighborhood (ϵ)" * of AB a broken line $\bar{\mathfrak{C}}$ joining A and B and

* Compare, for the definition of neighborhood, Osgood, *l. c.*, p. 106.

made up of segments having alternately the slope 0 and -1. For such a broken line the integral I is evidently equal to

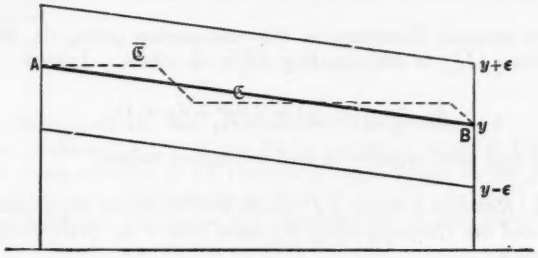

FIG. 1.

zero, the integrand being zero between the limits of integration, whereas for the line AB the integral I has the positive value

$$I_{\mathfrak{C}} = m^2(m+1)^2 \cdot (x_1 - x_0).$$

Case II: $m = 0$ or $m = -1$.

In this case the *line AB minimizes the integral I;* for it furnishes the value $I = 0$ whereas every other curve joining A and B and representable in the form $y = f(x)$ furnishes a positive value.

Case III: $m > 0$ or $m < -1$.

In this case the *line AB minimizes the integral I.* For the set of straight lines parallel to AB constitutes a "field of extremals" about AB, hence by Weierstrass's * theorem we have for every curve $\bar{\mathfrak{C}}$ not coinciding with AB, joining A and B and satisfying the conditions of § 1,

$$\Delta I = \int_{x_0}^{x_1} E(x, y, y', \bar{y}')dx, \tag{5}$$

where x, y is a point of $\bar{\mathfrak{C}}$; $\bar{y}'$ the slope of $\bar{\mathfrak{C}}$ at x, y; y' the slope at x, y of the extremal of the field passing through x, y, *i. e.*, $y' = m$; therefore

$$E(x, y, y', \bar{y}') = (\bar{y}' - m)^2[\bar{y}'^2 + 2(m+1)\bar{y}' + 3m^2 + 4m + 1],$$

* Compare Osgood, *l. c.*, p. 118; also Hilbert, *Archiv für Mathematik* (3) vol. 1 (1901), p. 231.

which is positive except where $\bar{y}' = m$. Hence it follows that

$$\Delta I > 0.$$

Our example illustrates in this case another point, viz., that *condition* (H_a) *is not necessary for a minimum.* Indeed

$$F_{y'y'}\big(x, f(x), p\big) = 2(6p^2 + 6p + 1),$$

which may take negative as well as positive values.

§3. *Relation between a Problem in Parameter Representation and the Corresponding Problem with* x *as Independent Variable.*

When one has become acquainted with Weierstrass's treatment of the simplest type of problems of the calculus of variations (viz., in parameter representation) one is inclined to regard the older method — in which x is taken as the independent variable — as antiquated and imperfect as compared with Weierstrass's method; unjustly however, for *the two methods deal with two clearly distinct questions* and which of the two deserves the preference depends upon the nature of the special problem under consideration.

When it is proposed to minimize the integral

$$I = \int_{t_0}^{t_1} F\left(x, y, \frac{dx}{dt}, \frac{dy}{dt}\right) dt, \tag{6}$$

where

$$F(x, y, kx', ky') = k\, F(x, y, x', y') \tag{7}$$

for every positive k, we consider a certain well-defined manifoldness $\mathfrak{N}$ of *curves*

$$x = \phi(t), \quad y = \psi(t),$$

out of which we have to select those curves which minimize the integral I.

If we consider, instead of $\mathfrak{N}$, another manifoldness $\mathfrak{M}$ of curves, we obtain an entirely different problem.

Let now $\mathfrak{M}$ be, in particular, the totality of those curves of $\mathfrak{N}$ which are representable in the form

$$y = f(x),$$

$f(x)$ being a single valued function of x. Then the new problem may be formulated thus: Among all curves (*functions*) $y=f(x)$ of the totality $\mathfrak{M}$ to find those which minimize the integral

$$I=\int_{x_0}^{x_1} F\left(x, y, 1, \frac{dy}{dx}\right)dx. \tag{8}$$

We distinguish the two problems as "the problem (T)" and "the problem (X)." Since $\mathfrak{M}$ is contained in $\mathfrak{N}$, it follows that *every solution of* (T) *which is representable in the form* $y=f(x)$ *is a fortiori also a solution of* (X); *but* (T) *may have solutions which are not representable in this form and which accordingly are not solutions of* (X).

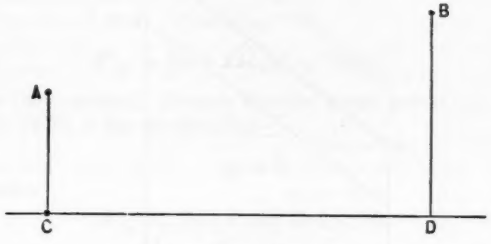

FIG. 2.

A well-known example of this kind is the so-called "discontinuous solution" of the problem of the minimum surface of revolution,* which consists of two perpendiculars AC and DB to the axis of revolution (the x-axis) and the segment CD of the latter. The broken line $ACDB$ is under certain assumptions concerning the positions of A and B a solution of the problem (T), but not of the problem (X), because the segments AC and BD are not representable in the form $y=f(x)$.

On the other hand, *the problem* (X) *may have solutions which are not at the same time solutions of* (T).

An example of this kind is furnished by the problem of § 2, case III. In this case the line AB furnishes, as we have seen, a smaller value for the integral

* See Kneser, Lehrbuch der Variationsrechnung, p. 179.

$$I=\int_{x_0}^{x_1} y'^2\left(+\,y'1\right)^2 dx,$$

than any other curve joining A and B, and representable in the form $y=f(x)$. But if we pass to the corresponding problem (T) and admit also curves which are not representable in this form, we can easily find neighboring curves which furnish still smaller values for the integral I than the line AB; for instance any broken line $\mathfrak{C}$ joining A and B and made up of

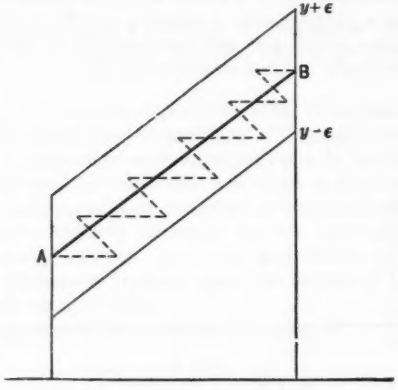

FIG. 3.

segments having alternately the slope 0 and -1. For such a broken line the integral is zero, whereas it is positive for the line AB. Hence AB is a solution of (X) but not of (T).

§ 4. *Example Illustrating the Insufficiency of Weierstrass's Condition in Case x is Taken as Independent Variable.*

Weierstrass discusses the problem to minimize the integral (6) under the assumption that $F(x, y, x', y')$, considered as a function of its four arguments is regular in the vicinity of every point $x=a$, $y=b$, $x'=a'$, $y'=b'$ for which a, b lies in a certain region of the xy-plane, while a', b' may have any system of values except $a'=0$, $b'=0$. Under this assumption he proves that the conditions corresponding to (I), (II′), (III′), (IV′) are *sufficient* for a minimum.

Nevertheless, for a minimum of the integral (1) under the assumptions stated in § 1, *the conditions* (*I*), (*II'*), (*III'*), *and* (*IV'*) *are not sufficient, not even if* (*IV'*) *be replaced by the stronger condition* (II'_a).*

This is shown by the following example:

Example II.—To minimize the integral

$$I = \int_0^1 (ay'^2 - 4byy'^3 + 2bxy'^4)dx, \tag{9}$$

a, b being two positive constants, with the initial conditions

$$y = 0 \text{ for } x = 0 \quad \text{and} \quad y = 0 \text{ for } x = 1.$$

Here Euler's equation reduces to

$$-y'' F_{y'y'} = 0, \tag{I}$$

where $$F_{y'y'} \equiv 2a - 24byy' + 24bxy'^2.$$

The only extremal through the two given points A: (0, 0) and B: (1, 0) is the straight line

$$\mathfrak{C}: \qquad y = 0.$$

Further

$$F_{y'y'}\big(x, f(x), f'(x)\big) \equiv 2a > 0. \tag{II'}$$

The set of extremals through A is the pencil of straight lines through A; hence there exists no conjugate point and condition (III') is satisfied.

Further

$$\frac{E(x, y, y', p)}{(p - y')^2} \equiv (a - 8byy' + 6bxy'^2) + p(-4by + 4bxy') + 2bxp^2.$$

Hence along $\mathfrak{C}$

$$\frac{E\big(x, f(x), f'(x), p\big)}{\big(p - f'(x)\big)^2} \equiv a + 2bxp^2 > 0. \tag{IV'}$$

* This statement seems to contradict directly the theorem on sufficient conditions given on page 118 of Osgood's article. The contradiction is, however, only apparent since Osgood makes the further assumption—see page 108—that "$F_{y'y'}(x, y, p)$ does not vanish in R" from which it follows that if $F_{y'y'}(x, y, p)$ is positive at all points of $\mathfrak{C}$ for every value of p, it is also *positive at all points* OF A CERTAIN NEIGHBORHOOD OF $\mathfrak{C}$, *for every value of* p.

The four conditions (I), (II'), (III'), (IV') *are therefore satisfied; even the stronger condition*

$$F_{y'y'}(x, f(x), p) \equiv 2a + 24bxp^2 > 0. \qquad (II'_a)$$

Nevertheless the line $\mathfrak{C}$ *does not minimize the integral* I.

For if we replace the line AB by the broken line APB, the

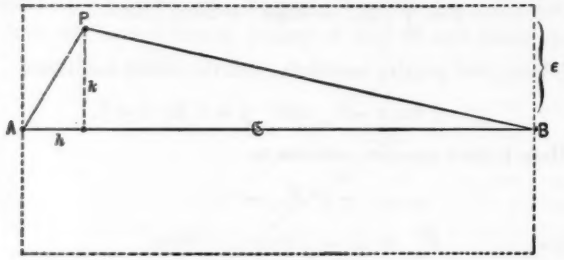

FIG. 4.

coördinates of P being $x = h > 0$ and $y = k$, the total variation of I is easily found to be

$$\Delta I = k^2\left[-\frac{bk^2}{h^2} + \frac{a}{h} + a + 3bk^2\right] + (h) \qquad (10)$$

where $$\lim_{h=0}(h) = 0.$$

Now let $\epsilon > 0$ be given as small as we please, then choose $|k| < \epsilon$ and let h approach zero, keeping k fixed. Then since $b > 0$ it follows that

$$\Delta I < 0$$

for all sufficiently small values of h, which proves that the line AB does not minimize the integral I.

The explanation of the fact that the four conditions in question are not sufficient in the one problem whereas the corresponding four conditions are sufficient in the other problem is as follows: In Weierstrass's problem no restriction is imposed upon the direction of the tangents of admissible curves, whereas in our problem the direction parallel to the y-axis, and no other direction, is excluded. In the former case the totality of admissible directions is closed, in the latter it is not closed.

UNIVERSITY OF CHICAGO,
June 13, 1902.

ON THE SUFFICIENT CONDITIONS IN THE CALCULUS OF VARIATIONS.

BY DR. E. R. HEDRICK.

(Read before the American Mathematical Society, December 28, 1901.)

THE sufficient conditions in the calculus of variations have recently received a great deal of attention; * and it would seem fitting that attempts be made to simplify their discussion whenever possible, and to render the agreement more exact between the known necessary and the known sufficient conditions. Such is the purpose of this paper, which also seeks to present the sufficient conditions in compact form. The work will to a large extent follow lectures delivered at Göttingen by Professor Hilbert, 1899–1901.

1. HILBERT'S INVARIANT INTEGRAL.† WEIERSTRASS'S SUFFICIENT CONDITION.

Let us consider a simple definite line integral

$$I = \int_{x_0}^{x_1} f(x, y, y')dx, \tag{1}$$

where

$$y' = \frac{dy(x)}{dx},$$

and where f is an analytic function of the three arguments x, y, y', in a certain region R. Let us then restrict ourselves to the consideration of curves of integration contained in a realm B, consisting of curves of the type

$$y = \phi(x), \tag{2}$$

* Du Bois-Reymond, *Math. Annalen*, 15; Kneser, Lehrbuch der Variationsrechnung, and many memoirs in *Math. Annalen*; Osgood, *Annals of Math.*, 2d ser., vol. 2, no. 3, and *Transactions Amer. Math. Soc.*, vol. 2, pp. 166, 273; Whittemore, *Annals of Math.*, 2d ser., vol. 2, no. 3; Bolza, Ithaca Colloquium (summer meeting, Amer. Math. Soc., Aug., 1901), unpublished, and *Transactions Amer. Math. Soc.*, vol. 2, p. 422; Weierstrass, lectures at Berlin, 1879-1882, unpublished; Hilbert, lectures at Göttingen, 1899–1901, unpublished, etc.

† Compare Osgood, *Annals of Math.*, *l. c.*, where Hilbert's proof is given.

where $\phi(x)$ together with its first derivative is a single valued and continuous function of x in the interval $x_0 \leqq x \leqq x_1$, and where $\phi(x_0)$ and $\phi(x_1)$ have certain fixed values y_0 and y_1 respectively. If the value of the integral I taken along a curve

$$y = y(x) \tag{3}$$

of the realm B be less than the value of I taken along any other curve of the realm B,

$$y = Y(x) \equiv y(x) + \eta(x), \tag{4}$$

where $Y(x)$ is restricted by the same conditions as y and also by

$$|\eta(x)| < \delta, \quad |\eta'(x)| < \delta, \quad x_0 \leqq x \leqq x_1, \tag{5}$$

δ being a positive constant chosen at pleasure, then the curve (3) is said to render the integral I a *weak* minimum, as compared to curves of the realm B. If in place of the conditions (5) we merely require that

$$|\eta(x)| < \delta, \quad x_0 \leqq x \leqq x_1, \tag{6}$$

then the curve (3) is said to render I a *strong* minimum.* If in addition to the condition (5) [or (6)] we require that

$$\begin{gathered}\eta(x_i) = 0\,; \quad x_0 + (i-2)\cdot\delta \leqq x_i \leqq x_0 + (i-1)\delta \leqq x_1\,; \\ (i = 2, 3, \cdots n, (n+1)\,;\ n\cdot\delta = x_1 - x_0),\end{gathered} \tag{7}$$

then we will say that the curve (3) renders I a *limited* weak [or strong] minimum.†

It can be shown ‡ that if the curve (3) is to render I a minimum (of any kind), $y(x)$ must satisfy Lagrange's equation

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0. \tag{8}$$

The solutions of this equation are called *extremals*.

Let us now assume: §

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* For the above, cf. Osgood, *Annals of Math.*, *l. c.*, p. 106.

† This is what Hilbert calls "ein Minimum bei stückweiser Variation."

‡ See Whittemore, *l. c.*, p. 130, where Hilbert's proof is reproduced.

§ Compare Osgood, *Annals of Math.*, *l. c.*, p. 113; *Transactions Amer. Math. Soc.*, *l. c.*, p. 168.

(a) That an extremal C,

$$y = y(x), \tag{9}$$

can be obtained, such that $y_0 = y(x_0)$ and $y_1 = y(x_1)$, *i. e.*, such that C passes through the two fixed points P: (x_0, y_0) and Q: (x_1, y_1) given in the problem.

(b) That a one parameter family of extremals

$$y = \phi(x, \alpha) \tag{10}$$

can be obtained, where ϕ is an analytic* function of x and α, $x_0 \leqq x \leqq x_1$, $|\alpha| < \epsilon$; and where

$$\phi(x, 0) \equiv y(x). \tag{11}$$

(c) That

$$\frac{\partial \phi(x, 0)}{\partial \alpha} \neq 0, \quad x_0 < x \leqq x_1.$$

It then follows that through any point of a suitably chosen neighborhood N about the curve C, one and only one extremal of the family (10) can be passed; and that (10) can be solved for α in terms of x and y,

$$\alpha = \psi(x, y), \tag{12}$$

where $\psi(x, y)$ is a singly valued analytic function of x and y, in the neighborhood N. Hence also

$$y' = \frac{\partial \phi(x, \alpha)}{\partial x}$$

can be expressed as a singly valued analytic function of x and y,

$$y' = p(x, y), \tag{13}$$

in the neighborhood N.

Hilbert now considers the integral

$$J = \int_{x_0}^{x_1} [f(x, y, p) + (y' - p) f_p(x, y, p)]\, dx \tag{14}$$

taken along any curve $\overline{C}$, $y = Y(x)$, joining P and Q, and

* Since f has been assumed analytic in x, y, y', it follows that the coefficients in (8) are analytic, and hence that the solutions of (8) are analytic.

lying in the neighborhood N, where y and y' refer to the curve of integration and p denotes the above function $p(x, y)$, and where f_p means $\partial f/\partial p$. It is then shown that this integral J is independent of the path of integration.* *i. e.*, independent of $Y(x)$. It is seen at once that $J_C = I_C$, where J_C denotes the value of the integral J taken along the curve C, and so on. But $J_{\bar{C}} = J_C$ since J is independent of the path. Hence the necessary and sufficient condition for a minimum, which is that $I_{\bar{C}} > I_C$, becomes at once $I_{\bar{C}} > J_{\bar{C}}$, or $(I - J)_{\bar{C}} > 0$, or

$$\int_{x_0}^{x_1} E(x, y, y', p)dx \Big]_{\bar{C}} > 0, \tag{15}$$

where C is any curve in N other than C, and where

$$E(x, y, y', p) \equiv f(x, y, y') - f(x, y, p) - (y' - p)f_p(x, y, p). \tag{16}$$

It follows that

$$E(x, y, y', p) \geqq 0, \quad x_0 \leqq x \leqq x_1 \tag{17}$$

for all x, y near the curve C, for the function p found above, and for any y' whatever, *is a sufficient condition for a strong minimum*,† where the sign of equality is to hold only along the extremals (10). And, moreover, the condition

$$E(x, y, y', p) \geqq 0, \quad x_0 < x < x_1, \tag{18}$$

for all x, y, p on C and for all y' whatever, *is a necessary condition for a strong minimum.*† For if, for simplicity, the family of extremals (10) be so taken as all to pass through $P: (x_0, y_0)$, then a curve can be found which renders the integral less than C does, in case $E < 0$ at any point of C in any direction $y' \neq p$. For we need only take as the comparison curve desired an extremal other than C, together with a curve which cuts C at this point in the given direction. The integral $\int E dx$ taken along such a curve is certainly negative.

Let us now consider a fixed point $R: (x, y)$, on the curve C, and $f(x, y, y')$ as a function of y' alone, for these fixed values of x and y. Let us then draw in a $y'f$-plane the curve

* See Osgood, *Annals of Math.*, *l. c.*, p. 122.

† Provided, of course, that conditions (a), (b), (c) hold.

$f=f(y')$. By (16) the inequality $E>0$ becomes

$$\frac{f(y')-f(p)}{y'-p} \gtrless f_p(p) \text{ according as } y'-p \gtrless 0. \tag{19}$$

Let θ be the angle made by the axis of y' with the tangent to the $y'f$-curve at the fixed point where $y'=p$. And let ψ be the angle between the y' axis and the chord joining this fixed point to a variable point where $y'=y'$. Then, as is easily seen from a figure,

$$\tan\theta = f_p(p) \quad \text{and} \quad \tan\psi = \frac{f(y')-f(p)}{y'-p}.$$

Hence the inequality (19) becomes

$$\tan\psi \gtrless \tan\theta \quad \text{according as} \quad y' \gtrless p. \tag{20}$$

But this is surely fulfilled (except, of course, for $y'=p$) provided the $y'f$-curve is always concave upward, the f axis having been taken vertical, for the range of values of y' considered.

Hence the condition

$$\frac{\partial^2 f}{\partial y'^2} > 0, \tag{21}$$

*for all x, y on the curve C and for all y' considered, is a sufficient condition for a minimum.** For if (21) is satisfied, the $y'f$-curve for every point (x, y), on *or near*† C, will be concave upwards, for the whole range of values of y' considered, and hence, by the above, the Weierstrass sufficient condition (17) will be satisfied.‡

The condition (21) is, however, *by no means* necessary. A necessary condition usually attributed to Legendre resembles it *in form*. This *necessary* condition requires that $\frac{\partial^2 f}{\partial y'^2} \geqq 0$ for all

* Weak or strong, according as the values of y' are restricted, or not. Here, as elsewhere, the region R of page 11 must not be overstepped. In particular, the value $y'=\infty$ must always be carefully investigated.

† On account of the continuity of $\frac{\partial^2 f}{\partial y'^2}$ in x, y, y'.

‡ This also follows from $\underset{y'=p}{\text{Limit}}\left[\frac{E(x, y, y', p)}{(y'-p)^2}\right] = \left[\frac{\partial^2 f}{\partial y'^2}\right]_{y'=p}$.

*x, y on the curve C and for $y' = p$.** A proof of this theorem is given in the following section.

2. Limited Variation. Legendre's Condition.

If we now consider limited variation only, we have the following theorem: *Lagrange's, Legendre's narrower,† and Weierstrass's ‡ conditions are together sufficient for a limited strong minimum.*

We suppose first found a solution C of Lagrange's equation connecting the points P and Q. Around any point of C a region can so be bounded off that one and only one extremal exists joining the given point to any point of the region; which follows at once from the Cauchy-Kowalewski existence theorems, since $\left.\frac{\partial^2 f}{\partial y'^2}\right]_C \neq 0$ at such a point, from the narrower condition of Legendre. If now we require that $\eta(x) = 0$ at least once in the interior of each of a set of regions such that the Cauchy-Kowalewski solutions starting from the one end point are unique up to the other end point, we surely have in each such region a satisfactory field for the Weierstrass-Hilbert theory. It follows that the Weierstrass-Hilbert reasoning can be applied, and hence, if the Weierstrass condition is satisfied, the curve C actually renders the integral I a limited strong minimum.

Likewise it is clear that *Lagrange's, and Legendre's narrower conditions alone are together sufficient for a limited weak minimum.* For Legendre's condition, $\left.\frac{\partial^2 f}{\partial y'^2}\right]_C > 0$ is satisfied. But this insures § the fulfillment of Weierstrass's sufficient condition, by the end of Section 1, since $\frac{\partial^2 f}{\partial y'^2}$ is a continuous func-

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* For this condition we shall use the notation $\left.\frac{\partial^2 f}{\partial y'^2}\right]_C \geqq 0$; and we shall call it, temporarily, Legendre's *broader* condition; later, Legendre's *necessary* condition. The corresponding condition, $\left.\frac{\partial^2 f}{\partial y'^2}\right]_C > 0$, where the sign of equality may not hold, we shall call, temporarily, Legendre's *narrower* condition; later, *sufficient*. It is readily seen that $\left.\frac{\partial^2 f}{\partial y'^2}\right]_C > 0$ is sufficient for a weak minimum, under conditions (a), (b), (c).

† See preceding footnote.

‡ See condition (17) above.

§ For weak variation, of course.

tion of x, y, y'. On account of this theorem we will henceforth call the condition $\frac{\partial^2 f}{\partial y'^2}\Big]_C > 0$ *Legendre's sufficient condition.*

We will now proceed to show that Legendre's broader condition is necessary for any kind of a minimum whatever. We have seen that Legendre's sufficient condition, $\frac{\partial^2 f}{\partial y'^2}\Big]_C > 0$ is, together with Lagrange's condition, sufficient for a limited weak minimum. But that I have a *minimum* is equivalent to saying that $-I$ has a *maximum*. Hence $\frac{\partial^2 f}{\partial y'^2}\Big]_C < 0$ is, together with Lagrange's condition, a sufficient condition for a limited weak *maximum*. Given now a solution of Lagrange's equation, C, joining P and Q, if $\frac{\partial^2 f}{\partial y'^2}\Big]_C < 0$ at any point of C, then $\frac{\partial^2 f}{\partial y'^2}\Big]_C < 0$ for a whole interval, since $\frac{\partial^2 f}{\partial y'^2}$ is a continuous function of x, y, y', and C is a continuous curve whose tangent varies continuously. But since $\frac{\partial^2 f}{\partial y'^2}\Big]_C < 0$ is a *sufficient* condition for a limited weak *maximum*, it follows that *any limited weak* variation in this interval gives us a curve for which I has a *less* value than along C. *It follows that Legendre's broader condition,* $\frac{\partial^2 f}{\partial y'^2}\Big]_C \geqq 0$, *is a necessary condition for a minimum of any kind whatever;* since otherwise we have actually found a comparison curve C' such that $I_{C'} < I_C$, and this comparison curve occurs in any possible choice of realms of curves to be considered *since it is a limited weak variation of* C, the tangents to which can be made, if necessary, to turn continuously.* On account of this theorem we shall henceforth call the condition $\frac{\partial^2 f}{\partial y'^2}\Big]_C \geqq 0$ *Legendre's necessary condition.*

It seems fitting at this point to call attention to the general usefulness of the idea of the limited variation, which has led us so easily to Legendre's necessary condition. The question

* Or even analytically, by the now well known method originally due to Schwarz.

of geodetics in the surface theory, and many questions in mechanics, notably the Hamilton-Jacobi theory, require only the use of *limited* variation, and indeed it is peculiarly suited to the needs of these and other similar subjects. Its greater simplicity in statement and treatment commend it above methods commonly employed, and its results are more easily obtained and grasped. For instance, in the surface theory, any geodetic line actually renders the integral of length on the surface a *limited* strong minimum along *any* portion of it which joins *any* two points upon it.* This statement brings into a clear light the essential characteristics of a geodetic as a "shortest" line, and its application to such special surfaces as the sphere and the anchor ring are particularly vivid.

3. Unlimited Variation. Production of Jacobi's Condition from Weierstrass's Condition.†

Jacobi's condition was originally erroneously stated as a sufficient condition, and was shown by Weierstrass to be insufficient for strong minima. Thus far we have obtained Weierstrass's condition (Section 1) as sufficient for a strong minimum under the necessary conditions (a), (b), (c) of Section 1. We now propose to obtain Jacobi's condition as a necessary condition for any (*unlimited*) mininum from consideration and elimination of these conditions (a), (b), (c). We shall then show that Jacobi's condition (together with Lagrange's and Legendre's) is a *sufficient* condition for *weak* minima, the Weierstrass condition being unnecessary. Thus the Jacobi condition precisely covers the introduction of the *unlimited* variation, for exactly analogous results have been obtained for *limited* variation (Section 2) with its omission.

We will suppose the Lagrange condition satisfied, since it is a *first* necessary condition for any discussion. Let the supposed solution C of our problem be the curve

$$y = y(x) \tag{1}$$

joining P: (x_0, y_0) and Q: (x_1, y_1). The Lagrange equation

$$y'' \frac{\partial^2 f}{\partial y'^2} + y' \frac{\partial^2 f}{\partial y \partial y'} + \frac{\partial^2 f}{\partial x \partial y'} - \frac{\partial f}{\partial y} = 0 \tag{2}$$

* In this particular theorem, of course, no new *fact* is involved.

† In connection with this section, see Osgood, *Transactions Amer. Math. Soc.*, *l. c.*, p. 166.

has in general a two-parameter family of solutions

$$y = \phi(x, c_1, c_2) \tag{3}$$

where ϕ is an analytic function of x, c_1, c_2, if, as supposed, f is analytic in x, y, y'.

Let us now suppose that c_1 and c_2 are so chosen that

$$\phi(x, 0, 0) \equiv y(x), \tag{4}$$

so that for $c_1 = 0$, $c_2 = 0$, our curve C results. Further let $c_1 = t\gamma_1$ and $c_2 = t\gamma_2$. Then (3) becomes

$$y = \phi(x, t\gamma_1, t\gamma_2). \tag{5}$$

Since ϕ is analytic in c_1 and c_2 we have, when γ_1 and γ_2 are regarded as constant,

$$\left\{\begin{aligned} y &= \phi(x, t\gamma_1, t\gamma_2) = \Phi(x, t), \quad \text{say,} \\ &= \Phi(x, 0) + t\frac{\partial \Phi(x, 0)}{\partial t} + \frac{t^2}{2!}\frac{\partial^2 \Phi(x, 0)}{\partial t^2} + \cdots \\ &= y(x) + t\phi_1(x) + t^2\phi_2(x) + \cdots \end{aligned}\right. \tag{6}$$

which converges uniformly for sufficiently small values of t.

Now, regarding γ_1 and γ_2 as constant, and (6) as our *one*-parameter family of solutions required by condition (b), Section 1, let us set

$$p = \frac{\partial \Phi(x, t)}{\partial x} = \frac{dy(x)}{dx} + t\frac{d\phi_1(x)}{dx} + t^2\frac{d\phi_2(x)}{dx} + \cdots \tag{7}$$

If now we can solve (6) for t as a single valued function of y and x, we can insert the value found in (7) and have p defined as a single valued function of y and x, *i. e.*, of the position; and this will hold for sufficiently small values of t, since the series used are then uniformly convergent, *i. e.*, it will hold for all points in a suitably chosen neighborhood of C. Such a solution will then satisfy the purposes of the Weierstrass-Hilbert theory. But for this (compare condition c, Section I) it is only necessary that

$$\phi_1(x) \neq 0. \tag{8}$$

But

$$\phi_1(x) = \frac{d\Phi(x, t)}{dt}\Big]_{t=0},$$

whence

$$(9) \qquad \phi_1(x) = \frac{\partial\phi(x, c_1, c_2)}{\partial c_1}\frac{dc_1}{dt} + \frac{\partial\phi(x, c_1, c_2)}{\partial c_2}\frac{dc_2}{dt}$$

$$= \frac{\partial\phi}{\partial c_1}\gamma_1 + \frac{\partial\phi}{\partial c_2}\gamma_2 = \eta_1\gamma_1 + \eta_2\gamma_2,$$

$$\text{where} \quad \eta_1 \equiv \frac{\partial\phi}{\partial c_1}, \quad \eta_2 \equiv \frac{\partial\phi}{\partial c_2}.$$

If then we can so choose γ_1 and γ_2 that $\phi(x)$ does not vanish between x_0 and x_1 the Weierstrass-Hilbert theory can be applied.

And further, the possibility of such a choice is necessary for the existence of a minimum. For, consider the family of extremals out of (3) which pass through (x_0, y_0) and suppose that γ_1 and γ_2 are so chosen that (6) represents this family.* If then $\phi_1(x)$ vanishes at any point R between x_0 and x_1 then the equations

$$(10) \qquad \begin{cases} y = \Phi(x, t), \\ \dfrac{d\Phi(x, t)}{dt} = 0, \\ y = y(x) \equiv \Phi(x, 0) \end{cases}$$

are satisfied at this point, since

$$\frac{d\Phi(x, t,)}{dt}\Big]_{t=0} = \phi_1(x).$$

Hence the envelope of the above family of extremals through (x_0, y_0) cuts the curve C at any such point as R.

If now this envelope be a single point, then compare the value I_C with $I_{\bar{C}}$ where $\bar{C}$ is any other extremal of the family. Since J is independent of the path and since J reduces to I for all extremals, it follows that C does not render I a proper (unlimited) minimum of any sort whatever, between P and R or

* It can be shown from the considerations at the end of this Section, that this is the best possible choice.

between P and any point beyond R. For $I_C]_P^R = I_{\bar{C}}]_P^R$ and $\bar{C}$ is an analytic weak variation of C.

If on the other hand the envelope is a curve, then compare $I_C]_P^R$ with $I_C]_P^{R'} + I_E]_{R'}^R$, where $\bar{C}$ is any other extremal of the family, E the envelope, and R' the point where $\bar{C}$ meets E. Since the envelope is tangent at every point to an extremal of the family, it follows, as before, that $J_E = I_E$ and hence $I_C]_P^R = I_{\bar{C}}]_P^{R'} + I_E]_{R'}^R$. Hence C does not render I a proper * (unlimited) minimum between P and R nor between P and any point beyond R.

It follows that if we can so choose γ_1 and γ_2 that

$$\phi_1(x) \neq 0, \quad x_0 < x \leqq x_1,$$

the Weierstrass-Hilbert theory can be applied; and further the possibility of such a choice is a necessary condition for the existence of an (unlimited) minimum of any sort.

We know that

$$y = \phi(x, c_1, c_2) = \Phi(x, t) \tag{11}$$

must satisfy Lagrange's equation

$$L(y) \equiv \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0 \tag{12}$$

or

$$0 = L(\Phi(x, t)) = L(y(x) + t\phi_1(x) + t^2\phi_2(x) + \cdots) \tag{13}$$

$$= L(y(x)) + \frac{dL(y(x) + t\phi_1(x) + \cdots)}{dt}\Big]_{t=0} t + \frac{d^2L}{dt^2}\Big]_{t=0} \frac{t^2}{2!} + \cdots$$

for all sufficiently small values of t. Hence the coefficient of t must vanish separately,

$$\frac{d}{dt}\{L(y(x) + t\phi_1(x) + t^2\phi_2(x) + \cdots)\}\Big]_{t=0} = 0. \tag{14}$$

But $L(\xi)$ involves $\xi, \frac{d\xi}{dx}, \frac{d^2\xi}{dx^2}$. Hence we have

* It can indeed be shown that C does not even render I an improper minimum, for the envelope cannot be an extremal. It is to be noticed that all the above work holds even when the extremals cut the envelope.

$$(15)\qquad \frac{\partial L(\xi)}{\partial \frac{d^2\xi}{dx^2}} \frac{d\frac{d^2\xi}{dx^2}}{dt} + \frac{\partial L(\xi)}{\partial \frac{d\xi}{dx}} \frac{d\frac{d\xi}{dx}}{dt} + \frac{\partial L(\xi)}{\partial \xi} \frac{d\xi}{dt}\Big]_{t=0} = 0,$$

where

$$\xi \equiv y(x) + t\phi_1(x) + t^2\phi_2(x) + \cdots;$$

or

$$(16)\qquad \frac{\partial L(y)}{\partial y''} \phi_1''(x) + \frac{\partial L(y)}{\partial y'} \phi_1'(x) + \frac{\partial L(y)}{\partial y} \phi_1(x) = 0,$$

an equation which must be satisfied by $y = y(x)$. Using the definition of $L(y)$, we find by a simple calculation that $\phi_1(x)$, and hence both η_1 and η_2, must satisfy the following linear differential equation of the second order:

$$(17)\qquad f_{y'y'} \cdot \eta'' + \frac{df_{y'y'}}{dx}\eta' + \left(\frac{df_{yy'}}{dx} - f_{yy}\right)\eta = 0,$$

where

$$\eta'' \equiv \frac{d^2\eta}{dx^2}, \qquad f_{y'y'} \equiv \frac{\partial^2 f}{\partial y'^2}, \qquad \text{etc.},$$

as usual. But (17) is precisely Jacobi's equation; and Jacobi's whole condition consisted in finding whether an integral of this equation

$$\phi_1(x) = \gamma_1\eta_1 + \gamma_2\eta_2$$

existed, which did not vanish between x_0 and x_1. Hence the above are precisely Jacobi's conditions, except that the equality sign in our condition must be omitted in the usual discussion, and the corresponding case made a subject of further investigation.

The rest of the results announced at the beginning of the article now follow readily, and will be left to the reader.

4. Summary of Conditions.

Collecting our results, we may say that the following conditions hold:

	Limited Variations.		Unlimited Variations.	
	Weak.	Strong.	Weak.	Strong.
Necessary.	Lagrange's, Legendre's necessary.	Lagrange's, Legendre's necessary, Weierstrass's necessary.	Lagrange's, Legendre's necessary, Jacobi's.	Lagrange's, Legendre's necessary, Jacobi's, Weierstrass's necessary.
Sufficient.	Lagrange's, Legendre's sufficient.	Lagrange's, Legendre's sufficient, Weierstrass's sufficient.	Lagrange's, Legendre's sufficient, Jacobi's.	Lagrange's, Legendre's sufficient, Jacobi's, Weierstrass's sufficient.

It is seen on glancing at the table that from the simple conditions (Lagrange's and Legendre's) for limited weak variation we proceed to any other case by adding Weierstrass's conditions in the case of a strong minimum, and Jacobi's in case of an unlimited minimum, only. It is to be hoped that advances may be made in bringing the necessary and sufficient conditions more closely together or into entire coincidence. The above table represents *substantially* the present known conditions, in the belief of the writer.

In special problems the irksomeness of these conditions can sometimes be circumvented. For instance, *given a problem in which* $\frac{\partial^2 f}{\partial y'^2} > 0$ *for all values of* x, y, y', *then the necessary and sufficient condition for a limited strong minimum is the possibility of finding a solution of Lagrange's equation joining the two given end points.* Such is the case in the geodetic problem and also in the integral which leads to Hamilton's principle; and in each of these cases, fortunately, a limited strong minimum is all that is desired. Similar simplification occurs in every case when $\frac{\partial^2 f}{\partial y'^2} > 0$ for *all* x, y, y'. For then Legendre's and Weierstrass's conditions are always satisfied, and may be abstracted from the above table. For this reason Hilbert has called a problem in which $\frac{\partial^2 f}{\partial y'^2} > 0$ for all x, y, y' contained in a singly connected region R, in which the given end points lie, a "*regular*" problem of the calculus of variations.

5. Conclusion. Hilbert's Existence Theorem.

A misunderstanding has sometimes arisen in regard to Hilbert's existence proof. Its purpose is not to establish further conditions nor to solve more general problems, but is to reassure us as to the possibility of satisfying the conditions already found, in the more favorable cases. We have seen that a *necessary* condition for any minimum was the possibility of finding a solution of Lagrange's equation joining the two end points P and Q; and that this is a *sufficient* condition for a limited strong minimum, in a *regular* problem. That such a solution exists, even in the most favorable cases, is by no means certain from the ordinary theory of differential equations. It was the original purpose of Hilbert's proof (merely) to show that, in this most favorable case of a regular problem, a curve must exist which renders our integral an unlimited strong minimum, compared with all continuous comparison curves; and that the minimizing curve is composed of a finite number of pieces of extremals. Hilbert's existence theorem may therefore properly be called a theorem in differential equations.

At present the results cannot be said to demonstrate more than that a curve exists which renders the integral an *improper* minimum. This result *cannot* be extended in general, and it remains to show that Lagrange's equation is necessary, not only for proper, but also for improper minima.

In conclusion, the author desires to enter protest against the extreme complication recently introduced in some quarters into the essentially simple subject of the calculus of variations. In the case of the only modern text-book on the theory,* this condition is so exaggerated as to essentially mar the usefulness of the book, in that many who would otherwise interest themselves in the subject, are repelled by the style and treatment. It is to be sincerely regretted, in the opinion of the writer, that this tendency has been followed in some of the recent memoirs.

Yale University,
June, 1902.

* Kneser, Variationsrechnung.

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Einführung in das Studium der theoretischen Physik, inbesondere in das der analytischen Mechanik, mit einer Einleitung in die Theorie der physikalischen Erkenntniss. Von P. VOLKMANN. Leipzig, B. G. Teubner, 1900. xvi + 370 pp.

Quelques Réflexions sur la Mécanique, suivies d'une première Leçon de Dynamique. Par EMILE PICARD. Paris, Gauthier-Villars et Fils, 1902. 56 pp.

FROM the standpoint of the teacher who has to face the problem of introducing students to the elements of mechanics both theoretical and practical, probably the most interesting and suggestive work which has appeared during the last few years, is that of Dr. August Föppl, professor at the school of technology (Technische Hochschule) at Munich. The books are especially noteworthy because Professor Föppl teaches those who are to be engineers — practical men — and yet at the same time insists on teaching in such a manner that his pupils shall acquire not the applications alone but a firm grasp of the general theoretical principles which underlie those applications. This method of instruction unfortunately may not appeal to some in this country who care only for what our engineers can do and not at all for what they may know. But certainly it is much better to teach the thought and theory in addition to the routine and practice, or in conjunction with it. In fact, as Professor Föppl points out, the engineer can not be thoroughly master of his work and capable of handling whatever new difficulties may arise unless he knows the why as well as the how.

What then is the sort of lectures that a thorough scientist finds feasible when addressing a class of engineering students at one of the leading German scientific schools? This question

* Continued from the June number of the BULLETIN.

may be answered by looking into Professor Föppl's four volumes —for happily the author has followed his lectures closely. That the work is found practical and acceptable in Germany is shown clearly by the demand for a second edition within about two years of the time of first publication. In this review we shall lay especial stress on the first and fourth volumes, which are less technical than the second and third and which consequently are more interesting to us here.

The first volume, Introduction to Mechanics, discusses the elementary portions of the kinetics and statics of particles and rigid bodies with or without friction, of the theory of elasticity, and of hydromechanics. Throughout, the object of the author is to give his pupils a general knowledge of mechanical theory such that later the more advanced or special branches will be seen to fit on in their proper places. The language is natural and simple. The only mathematical prerequisite is a fair knowledge of differential and integral calculus. Every attempt is made to relieve the reader from difficulties of presentation in order that he may concentrate his undivided attention upon the new difficulties of the subject itself.

There is one possible exception to this statement. The author throughout all four volumes uses vector analysis wherever it is appropriate. The notations are those which he has previously employed in his Einführung in die Maxwell'sche Theorie der Electrizität. The vectors are printed in heavy faced German type. If we may be permitted to change to English Clarendons, let $\mathbf{a}$, $\mathbf{b}$ represent two vectors and $\mathbf{e}$ a vector of unit length normal to $\mathbf{a}$ and $\mathbf{b}$ upon that side of the $\mathbf{ab}$-plane upon which rotation from $\mathbf{a}$ to $\mathbf{b}$ appears counterclockwise. Then the scalar and vector products of $\mathbf{a}$ and $\mathbf{b}$ are defined as

$$\mathbf{ab} = -S\mathbf{ab} = \mathbf{a}\cdot\mathbf{b} = [\mathbf{a}|\mathbf{b}] = ab\cos(\mathbf{a},\mathbf{b}),$$

$$V\mathbf{ab} = V\mathbf{ab} = \mathbf{a}\times\mathbf{b} = [\mathbf{ab}] = ab\cos(\mathbf{a},\mathbf{b})\,\mathbf{e}.$$

(The notations are those of Föppl, Sir W. R. Hamilton, Gibbs, and Grassman in order.) Although the author had found in his own experience that vector methods were better suited to pedagogical purposes than cartesian analysis, he had grave fears at first lest the public at large, especially in Germany, should disagree with him and dislike his method. Yet he went boldly ahead and published it. The result was a little surpris-

ing to him. Those who used the volumes were pleased, and even the reviewers could not make objections without adding the qualifying statement that: "The presentation of the chief general theorems of mechanics in terms of vectors possesses a simplicity and intelligibility which could not be obtained by the old method of cartesian coördinates." The great popularity of Professor Föppl's work seems to bear out this judgment and to indicate further that the use of vectors certainly does not increase the difficulty of learning mechanics.

What style shall be adopted in presenting elementary mechanics is one of the most difficult questions an author or a teacher must meet. The students' previous training in algebra, trigonometry, analytic geometry, and calculus as it is generally taught has been necessarily quite formal. These mighty algorithms of formal mathematics must be learned so that they may be applied with readiness and precision. But with mechanics comes the application of these algorithms and formal do-by-rote methods, though often possible, yield no results of permanent value. How to elicit and cultivate thought is now of primary importance. There are two opposite methods. First. Give the student a syllabus of the subject in which the general theory, the bare facts, are presented with as little discussion as possible. The student must then supply his own discussion, his own details. He must think the subject over until he sees its meaning. If he has patience, ability, and a little guidance the result is good. Otherwise he gets little or nothing. Second. Enter into a discussion of the subject, touch on this side and on that, show what is known and what is not, point out where definitions are needed, where experiment must be appealed to, and thus by merely dwelling on the different aspects of the subject allow the true significance of it all to become apparent to the student. This method is often very successful in introducing new ideas such as mass, time, and force. It is practically taking them as innate ideas after talking long enough to insure that this innateness has been recognized. But it also has its disadvantages. For unless it is well handled the student is apt to be bewildered by the lack of definiteness, and to lose faith in his ability to work accurately. It may be remembered that we spoke favorably of Professor Gray's Treatise on Physics in that he struck a mean between these two methods.

Professor Föppl, writing for a totally different class of readers,

makes an uncommonly successful use of the second — the discursive method. The discussion is half historical, half scientific, such as is found much more elaborately worked out in Professor E. von Mach's Science of Mechanics, and is sure to appeal to one who commences mechanics with his common sense alert. The introduction to Volume I, a short essay on the origin and aim of mechanics, is excellent diet for future engineers and physicists. The author points out what is meant by experience — true, useful experience — and shows how the fact that geometry was given a definitive, logical form as long ago as the time of Euclid may be interpreted not as proving that geometry is not ultimately founded on experience, but rather as indicating how long the experimental stage antedated Euclid. To give mechanics a definitive form is now possible, as the work of Hertz and others shows. Some say that this straightforward logical method of presentation is the *simplest*. That depends on what is meant by *simple*. From the standpoint of the learner the historical crude method appears easier than the logical. Here Professor Föppl touches either rightly or wrongly on a great pedagogical principle. Perhaps it is interesting and not unsuggestive to note that while teachers and authors in our universities are trying to give calculus a logical rigor and mechanics a definitive set of axioms, there is in the meantime a great commotion among teachers of elementary geometry as to whether this logical definitive method of Euclid is at all suitable for beginners. Some scientists of note, Professor John Perry and Mr. Oliver Heaviside, for example, have taken part in the discussion and placed themselves emphatically upon the negative. As far as mechanics is concerned Professor Föppl seems to agree with them.

After the introduction, twelve pages of the text are devoted to general discussion of the ideas involved in particle, inertia, and force. The author does not define; he prefers to discuss. His treatment of a particle is particularly noteworthy. On page 17 we read: "The bodies or parts of bodies which we shall regard as particles may be reasonably small, but must not be very (*i. e.*, unlimitedly) small." The earth is a particle for many problems; a molecule for few. Setting a lower limit for the size of a particle in dynamics is usually forgotten; yet, sooner or later, it is of the greatest importance if students of atomic theories are not to be confused. A particle must generally be regarded either as composed of a large number of molecules or as a small part of an atom.

We might point out further that the common definition of density at a point as

$$D = \frac{dm}{dv}$$

is quite meaningless even in the case of a so-called homogeneous body unless we make an assumption which is never physically correct and which need never be made, namely that mass is a continuous, not to say differentiable function of position. Might it not be better to tell the student, especially one who is to be a physicist, that density and a large number of physical quantities are averages — averages which are taken over a region so large in comparison to the phenomenon averaged that the result is sufficiently stable to be considered as a definite number for practical purposes, but over a region so small with respect to the total region in which the phenomenon exists that the methods of the differential calculus may be employed in expressing and handling the average? Whether it is better to make physical hypotheses which are ultimately untenable in order that in the meantime mathematics may be applied with absolute rigor or to state that the methods of pure mathematics, though strictly not applicable, yield results which within negligible errors agree with the phenomena observed or observable? Professor Föppl seems to believe that discussion made in the right spirit is, all things considered, more satisfactory to practical men than precise definitions made on impossible physical hypotheses.

The law of inertia is explained and illustrated by Foucault's pendulum experiment. The illustration is subtle but striking. The fact that the pendulum will beat on in its own absolute plane regardless of the earth is particularly impressive and quite intelligible to students. Force is then discussed qualitatively; but it is only after the treatment of the motion of freely falling bodies that quantitative definitions of force and mass are given. Experiment is called upon to establish the fact that the acceleration g, due to gravity, is constant at a particular point in the surface of the earth. The force which the earth exerts on a body is called the weight of that body. Weights may be compared and a unit adopted. Mass may be introduced by means of the equation

$$W = mg = m\frac{d^2x}{dt^2}.$$

Finally by making the assumption that any force is essentially like the force due to gravity in that it may be expressed numerically in terms of weight units the author arrives simultaneously at a measure of force in general and at the fundamental dynamical equation for motion in one dimension—that is, the expression

$$F = ma = m\frac{d^2x}{dt^2}$$

Thus far only rectilinear motion has been treated. The author now proceeds to motion in two or three dimensions, to the parallelogram law, and the principle of the superposition of different motions. The discussion accorded to centrifugal force is perhaps unnecessarily long and is concerned mainly with showing how that force should be regarded as the equal and opposite reaction to the centripetal force, not by any means as a fictive force. Work and energy are introduced and the fundamental theorem that the work done is equal to the change in kinetic energy is proved. The author introduces the principle of virtual velocities and proves it from the parallelogram law. He states, however, that the inverse method of procedure is equally admissible. The principle of virtual velocities may be taken as the fundamental physical axiom instead of the parallelogram law which may then be proved from it. Finally a long treatment of moments and the solution of a number of problems complete the discussion of the dynamics of a particle —the first division of the first volume.

The second division deals with the mechanics of rigid bodies. After elementary kinematical considerations, statics and the principle of virtual velocities are taken up. This part of the work has been regarded with great disfavor by some reviewers, notably by Dr. K. Heun in the first number of the *Zeitschrift für Mathematik und Physik* for the current year. The statement which gives trouble is found on page 143: "Let us give to a body an arbitrary virtual displacement. This displacement can even be finite. It is, however, customary to apply the principle of virtual velocities in the case of a rigid body only to infinitesimal changes of position and we shall consequently derive it for such. The transference to a finite displacement offers no difficulty inasmuch as each finite displacement can be reduced to a sum of infinitesimal changes of position." If by "can even be finite." the author means can in *all* cases be finite,

as Dr. Heun seems to imagine, the statement is certainly in gross error; for it makes the principle of virtual velocities applicable only to cases of neutral equilibrium. But we see no reason for not interpreting the statement as meaning can in *some* cases be finite. In fact we are wont to use the principle of virtual velocities, as far as practical applications are concerned, mainly for finding the mechanical advantage and efficiency of machines. When a machine such as an inclined plane, a screw, or a pulley system is working at a constant rate the forces acting upon it must be in neutral equilibrium. We are inclined to believe that in a larger number of the problems in which virtual velocities are useful to the engineering student the forces must likewise be in neutral equilibrium. We might suggest that the author express himself a little more clearly on this point for the sake of those who seem to have misunderstood him. There are other points upon which the author might easily be defended from his reviewers if necessity made a defense advisable.

Before leaving the principle of virtual velocities we wish to ask whether the introduction of that subject into elementary mechanics is to be commended. Almost all English and some continental books practically ignore it. When it is not ignored the treatment given it is usually indefinite and unsatisfactory (Professor Föppl's book forms no exception) owing to the fact that the necessary and sufficient conditions as to rigidity, constraints, internal and frictional actions which must be satisfied in order that the principle be applicable are not clearly stated.

The third division of the volume deals first with the statical, second with the dynamical properties of centers of gravity, and finally with the kinetic energy of a rigid body. From this point of departure the fourth division leads naturally through the undissipated transformation of energy to the fifth division which is a long and practical treatment of frictional or dissipative forces. Sliding, rolling, axial, and string or belt frictions are each discussed carefully with applications.

In the sixth division the student is made acquainted with some of the fundamental ideas of elasticity and strength of materials which he will use in the third volume; in the seventh, with the direct and oblique impact of a perfectly elastic or perfectly inelastic sphere on another sphere or on a plane; in the eighth, with those qualitative and simpler quantitative facts of hydromechanics which are deducible without a great knowledge

of either mechanics or mathematics. It might have been better if Professor Föppl had included in the seventh division a discussion of imperfectly elastic impact. There is plenty of room as the volume contains only about four hundred pages.

The second volume, Graphical Statics, is about four hundred and fifty pages long and is divided into seven parts: composition and resolution of forces acting at a point or in a plane, the funicular polygon, forces in space, trusses in a plane, trusses in space, elastic deformation of trusses and statically undetermined trusses, theory of arches.

The inclination toward theoretical in addition to practical considerations is indicated in the treatment given to such subjects as the reciprocal relations between force-plans, the differential equations of a funicular curve under a load anywise distributed, the null system, the long analytic discussion of the theorem that a truss which contains only the necessary number of members and is stable must be statically determined and conversely is stable if it is statically determined for all possible loads. We miss, however, both in this volume and in the next, the interesting theorem that under any given conditions the most economical truss is one which is statically determined. We also happily miss the elaborate plates which generally accompany a work on graphical statics and which add so enormously to the price. The figures Professor Föppl gives are simple, comparatively inexpensive, and incorporated in the text. The object of the author is to give the reader a usable figure and allow him to draw one more accurate if he desires. Students of theoretical mechanics would do well to learn as much graphics as is contained in this work.

The third volume, Resistance or Strength of Materials, is also for the most part as instructive to those interested in theoretical mechanics as to engineers: for the author has not filled his book with masses of detail and tables of constants, but has dealt with the deduction of those theoretical results which must be obtained before the tables are usable. The eleven divisions are entitled; general investigations on stress, elastic deformation and specification of materials, bending of a straight bar, energy of deformation, bars with a curved line of center, bars on a yielding foundation, strength of plane plates supported along their contour, strength of vessels under external and internal pressures, torsional strength, buckling strength, and the elements of the mathematical theory of elasticity. The tendency of this

volume, like the others, to develop the theory of the subject would probably be distasteful, if not unintelligible, to our engineers in this country.

Owing to the nature of the subject exact mathematical developments, even when possible, would lead to expressions too complicated for use. Professor Föppl therefore introduces approximations very freely, and in many cases discusses the maximum error due to these approximations. There is one case where we fear his assumptions may not be quite justifiable. In solving the problem of a torsion of a rectangular beam he assumes that the strain is expressible by the first three terms of a power series. He omits, however, to test his results by means of the stress equations. The complete solution is due to St. Venant and is expanded into a series of transcendental terms. The last division gives a satisfactory and not very difficult introduction to the general mathematical theory of elastic isotropic bodies. This is practically the only elaborate addition to the lectures actually delivered by the author to his students. Even this would be given in the regular course, he states, were it not for sheer lack of time!

With the fourth volume, Dynamics, the author returns to more purely theoretical discussions. In fact, the first and fourth volumes, with the last division of the third, form a complete treatise on the elements of mechanics of a particle, a rigid body or a system of rigid bodies, of fluids, whether perfect or not, and of elastic bodies. The titles of the five divisions of this last volume are: dynamics of a particle, dynamics of a rigid body and a swarm of particles, relative motion, dynamics of systems of bodies (Lagrange's equations, etc.), hydrodynamics. The discussion is mainly a completion and extension of that in the first volume.

In the first division, central motion, potential, and harmonic vibrations are treated. The fact that the author does not overlook forced vibrations is noteworthy. Electrical developments of the last fifteen years have caused forced vibrations to grow constantly more and more important. Yet in few, even of the recent text-books, is any mention made of the subject. The treatment of the conservation of moment of momentum in the second division is enlivened by such interesting examples as the fall of a cat, the shrinkage of the earth, the ability to accelerate one's motion while swinging, and the effect on the motion of a ship due to the rotation of the machinery. The motion of a

top, impulses applied to a rigid body, and the theory of oscillations are some of the other subjects treated. The third division offers two proofs of Coriolis's theorem, and discusses some applications of relative motion.

In the fourth division that too great carelessness in stating restrictions, which was mentioned earlier in connection with the presentation accorded to the principle of virtual velocities in the first volume, obscures and mars the development of Lagrange's equation and Hamilton's principle. The method in which generalized forces are introduced is hardly the best, hardly convincing. A discussion of mechanical similarity and its applications closes the division. The treatment of hydrodynamics, like that of the mathematical theory of elasticity, is a valuable introduction to the subject. Perhaps they both are more nearly abstracts than introductions. For this includes, in addition to Euler's equations, the motion of a sphere through a perfect fluid, Helmholtz's theorem, the equations of a viscous fluid, and that contains not merely the fundamental equations, but also to a considerable extent the theories of Boussinesq, Hertz and St. Venant.

In conclusion let us state that the value of each volume is materially enhanced by the summary which concludes it and contains a list of the principal formulæ. The third volume has already been published by Gauthier-Villars in a French translation. We could wish that the entire work might be translated into English with the hope that it might stimulate our teachers and students of engineering to be more thorough and complete in their theoretical considerations. But it is doubtful whether the venture would prove much of a financial success. Engineers would think the work insufficiently practical. Physicists would think it too practical. In differing with the former we should call attention to that suggestive little book "Made in Germany," which was published not long since in England for the purpose of setting forth the reasons why Germany was coming so rapidly to the front in all lines of practical endeavor. One of the chief reasons given was the German's thoroughness and determination to know his field from beginning to end. In differing from the physicists we should point out that in those two volumes — the second and third — which they would probably scorn most, the name of Maxwell appears frequently. The presence of that name might aid in convincing students of theoretical and practical physics that they could

doubtless find quite as much profit in becoming acquainted with the contents of these volumes as in examining the more abstruse and unpractical parts of their science.

Professor Volkmann's Introduction to Theoretical Physics is a protest against the tendency toward the more or less artificial presentations given to mechanics by Kirchhoff, Hertz, and even Boltzmann. It is an attempt to show how by harking back to Newton and the English method of treating mechanics a greater clearness, a no less deep insight, and with sufficient care a no less scientifically accurate presentation may be obtained. The whole work is such a curious medley of history, philosophy of scientific method, foundations of mechanics, theoretical mechanics, and applications that we are at a loss to know for what sort of readers it can be intended. For those who already do know the subject the applications are not necessary; for those who do not, the philosophy is confusing. The author's explanation to his readers that "You have already busied yourselves to a certain extent with physics and mechanics and thereby commence these lectures with certain ideas and expectations which may be just as much of a help and value as of a hindrance and harm to you," seems on the whole to be of little use in trying to decide who his readers may be. We fear Professor Volkmann has fallen into an altogether too common German habit of writing to please himself.

The first division of the book is an introduction to the theory of physical knowledge and methods of physical thought. There are forty-four pages of octavo text with numerous footnotes referring to Newton, Kant, Lotze, Thomson and Tait, Helmholtz, and especially Volkmann. Then follows a bibliography of twenty-six titles of which ten are to previous articles by the author. To criticise the author's theory which is itself presented in compact form would take too much space. The theory is interesting to read; but how valuable it may be to either a theoretical or practical physicist is not clear.

The second division, about seventy pages, systematically lays the foundations for mechanics according to Galileo and Newton and works out the application to the case of a particle. The author postulates a conception of time, mass, and a universal ether which is to act as a fundamental system of reference. Just what these conceptions are we are unable to determine; but the necessity of postulating them apparently becomes evi-

dent after reading the author's discussion, if indeed it were not evident before. Next, the ideas of inertia, vector velocity, vector accelerations, and physical dimensions are introduced and applied to the motion of a projectile in vacuo.

As a foundation for kinetics, Newton's laws are stated and discussed critically. The analytic statement of the laws and the development of the dynamics of a free or constrained particle complete this division of the book. We are unable to ascertain precisely what a particle (Massenpunkt) is. The term does not appear in the index and a definition seems lacking in the text. Fortunately, however, we are told in due season that the postulates of space, time, and mass are: There is a *space* of entirely determinate geometric characteristics in which reality transpires —the euclidean space. There is a *time* which flows on uniformly and is independent of all phenomena. There is a quantity—the *mass*—which preserves its identity unchanged through all changes of phenomena. The word phenomenon has been used in its etymological meaning as a translation for Erscheinung.

The third division develops the mechanics of a system of particles whether the distribution be discontinuous or continuous. Especial attention is given to the general theorems of motion of the center of gravity, rate of change of moment of momentum, and energy. The student ought to be able to get a firm grasp on these fundamental principles and be ready to understand their application to those various vibrating physical instruments which are discussed theoretically in the fourth division.

In the fifth division, after twenty pages of preliminary remarks on fluids and how to treat them, the author settles down to hydrostatics first in its usual aspect of phenomena on the interior of a fluid and then in that part of the subject which takes into account the effects of surfaces of discontinuity and is more often known as the theory of capillary action. The sixth division contains such questions introductory to geophysics as the attraction of spheres, the relation of gravitation and weight, methods of determining the gravitation constant G and the density of the earth.

The discussion of the general principles of mechanics forms the seventh and last division of the work. The author had here an opportunity to collate and compare the various general principles—those of D'Alembert, Lagrange, Gauss, Maupertuis, and Hamilton. This opportunity he accepted. The result can

hardly be considered any material improvement over other treatments whether in definiteness or completeness. The volume ends with an index which is an absolute necessity for a book of such varied propensities.

If we have given an intelligible account of the contents of Professor Volkmann's work we shall feel satisfied. To review it connectedly is next to impossible. We had hoped that we might find in it a careful, precise, intelligible and connected treatment of theoretical physics or analytic mechanics. The precision and connection, at least, are for the most part lacking. This very lack, however, furnishes to those who are acquainted with mechanics, a stimulus to thought and thus the book may become more interesting and suggestive than one less indefinite —which is about as much as we can say for it.

Professor Picard's little volume consists of two parts — an essay on Mechanics and the Science of Energy and a First Lesson in Dynamics. The former is merely an extract from the general report on science which the author wrote in connection with the Exposition in 1900. The latter was delivered in 1894 and has hitherto been accessible only in periodicals. That they are now printed together ought to be a cause for congratulation among students and teachers of mechanics.

The first noticeable feature of the work is the perfect, the transcendent clearness of the style. It is unnecessary to inquire for whom the book was written. Any one can read it with ease, pleasure and profit.

The author shows how the idea held at the end of the eighteenth century that mechanics had attained a definite form has proved illusory, and how modern scientists, especially Hertz, have been able to reduce mechanics to a more logical set of axioms and a more entirely deductive form. Yet clearly behind a great admiration for Hertz and his work there stands out the evident feeling on the part of the author that something more practical, more anthropomorphic is quite as convenient and valuable.

There is one point which seems a trifle misleading. Upon page 14 we find the statement that dynamics of a particle will probably not become obsolete for the reason that atomic hypotheses play and will perhaps always play a preponderating rôle in science. The reason is not convincing. When an atom is treated as a definite object, it is not necessarily and not usually

regarded as a particle. It is either a rigid or a fluid body. A particle, as was pointed out above on page 28 of this review, may be a large aggregate of atoms or a small part of an atom, but not often the atom itself.

In his discussion of what is meant by the possibility of a mechanical explanation of all natural phenomena, Professor Picard points out that some mean the possibility of reducing the phenomena to one type of differential equations or another; some, the existence of a universally applicable principle like that of least action; and some, the construction of a dynamical model, as an example of which Kelvin's model of the ether may be mentioned.

The chapter on the science of energy states the different points of view of two classes of scientists, of which one believes energy to be nothing else that an abstract conception without real existence and of which the other believes in the real existence of different kinds of energy commutable into each other. Particular attention is given to the relation of known experimental results to the various laws of energy. Mention of the law of dissipation affords an opportunity to discuss whether or not that law is compatible with a mechanical explanation of the universe.

Professor Picard's first lesson in dynamics contains numerous departures from other introductory presentations of the subject. His particle is something small enough to be regarded without negligible error as a geometric point. This seems unnecessarily indefinite and yet unnecessarily definite. The important thing is that the motion does not differ appreciably from translation. Whether the particle may be regarded as a point has little to do with it. Inertia is presented in the Newtonian manner. Force is the name for any action which causes motion.

With the introduction of the measure of force there appears the first novelty. The author commences with a *field of force*. In particular, a *constant* field of force is such that particles left to themselves at different points of the field describe congruent paths in the same manner. The acceleration in a constant field is constant and may be taken as the measure of the field. Constant fields, or any fields at a definite point, may be compounded according to the parallelogram law. If force be defined statically by means of any dynamometer, experiment will show that the static and dynamic measures of force are proportional. In the same field, for example in the field of gravity

near the earth's surface, the forces acting on two masses are proportional to the masses. The unit of force or mass may therefore be chosen arbitrarily. The lesson closes with the determination of the fundamental differential equation of motion for a variable force and the discussion of work done by a force.

The mental acumen of a student who can understand and appreciate the whole of this, his first lesson, must be considerable. No further comment is necessary on this presentation of the fundamental ideas, which seems sufficiently accurate and modern, yet quite practical, and withal not far from Newton's classic treatment.

We have now passed in review about a dozen recent volumes on mechanics. Many different standpoints are represented, and often represented very well. But what is more noteworthy and valuable is the steady general improvement over older works, not only in the matter presented, but in the method of presentation. There is yet much to be done in producing treatises on mechanics. Perfection has not been reached, nor even nearly enough approached. There is one field in particular in which a thorough and careful work is greatly to be desired. That is the field of general dynamical theory. Sooner or later — let us hope it will be sooner — there must appear a work which shall take such general principles as D'Alembert's, Lagrange's and Hamilton's, and state just what are the restrictions under which each may be applied in practice. For instance, in Lagrange's Mécanique Analytique we find at least four distinct ways of writing the general equations of motion. Each has its peculiar advantages and restrictions. Who will set these forth? There is also plenty of room for a brief, almost syllabic, treatise on dynamical theory. Few students are able to appreciate how extremely limited purely theoretical considerations are in dynamics. Six or eight hours is quite sufficient to carry a lecturer from the beginning through the derivation of Lagrange's equations and Hamilton's principle, if all applications be omitted. And it is only thus that a thoroughly comprehensive grasp of dynamics can be obtained.

EDWIN BIDWELL WILSON.

YALE UNIVERSITY,
June 10, 1902.

ON A NEW EDITION OF STOLZ'S ALLGEMEINE ARITHMETIK, WITH AN ACCOUNT OF PEANO'S DEFINITION OF NUMBER.

Theoretische Arithmetik. Von O. STOLZ (Innsbruck) und J. A. GMEINER (Wien). I. Abtheilung: *Allgemeines. Die Lehre von den rationalen Zahlen.* Leipzig, B. G. Teubner, 1901. iv + 98 pp.

THIS publication of about 100 pages is the first instalment of a new and revised edition of Stolz's Allgemeine Arithmetik (1885–86), — a work which has long since proved indispensable to all who desire a systematic and rigorous development of the fundamental elements of modern arithmetic.

The revision thus far completed (probably about one seventh of the entire work) covers the first four chapters:

I. On quantities (Grössen) and operations (Verknüpfungen) in general.

II. On the natural numbers and the four fundamental operations.

III. On the general properties of any direct operation (Thesis, $a \circ b$) and its inverse (Lysis, $a \sim b$), as deduced from certain fundamental formal laws; in particular, the analytic theory of (absolute) rational numbers.

IV. On the synthetic theory of (absolute) rational numbers, with a treatment of systematic fractions.

The remaining chapters in the first part will contain (if the order in the old book is preserved) the theory of negative and irrational numbers, with an account of euclidean "ratios," followed by an elaborate treatment of the theory of limits as applied to functions of a real variable and to infinite series of real terms. The second part will then contain the theory of operations on complex numbers, including chapters on infinite series, infinite products and continued fractions. The complete work will belong to the Teubner series of mathematical text-books.

As one turns the pages of the new edition one is struck first of all by the great improvement in the general appearance of the book. The title itself, "theoretical arithmetic," is much

more lucid than the old title of "general arithmetic," which to the uninitiated might mean common algebra; the notes, historic and other, which used to be difficult of reference, are now placed conveniently as footnotes on the proper pages; and best of all the paragraph numbers in the several sections are now placed at the top of each page. All these changes will be appreciated by those who have had to refer often to the older book. The addition of some thirty problems or exercises is also very welcome.

In the body of the text there are numerous changes which greatly enhance the clearness of the presentation. The introduction of the rational numbers (III, 7) and the opening paragraph of IV should be especially mentioned. We notice that in III no attempt is made to prove the mutual independence of the postulates $A) - E)$, and as a matter of fact if E, 2) is made to read: "when $b > b'$ then $a \circ b > a \circ b'$" the postulate D_1 can be at once deduced as a theorem.

The only radical change from the plan of the first edition, however, is found in the second chapter, where the treatment of the natural numbers is now based on Peano's definition. The details of Peano's theory are here for the first time made accessible to readers unfamiliar with his mathematical language. Since, however, no systematic treatment of Peano's method is obtainable in English, it may be not without interest to devote the remainder of our space to a connected account of its principles.

Peano's Theory of the Natural Numbers.*

The fundamental concepts which form the starting-point of Peano's theory are three: (1) a class or assemblage, denoted by N; (2) a special object, denoted by E; (3) what we may call a "rule of succession," (denoted by $\circ$), according to which every object, a, of the given class N determines uniquely another object, $a\circ$, called the successor of a.

The nature of the assemblage N, of the special object E, and of the rule of succession $\circ$ is left wholly undetermined, except for the following five restrictions, or postulates:

* G. Peano: Arithmetices principia nova methodo exposita, Turin, 1889; "Sul concetto di numero," *Rivista di Matematica*, vol. 1 (1891), pp. 87–102, 256–267; *Formulaire de Mathématiques*, vol. 2 (1898), no. 2, pp. 1-59; vol. 3 (1901), pp. 39–44. Where I have written N and E, Peano uses N_1 and 1 or later N_0 and 0; and for $a\circ$, Peano uses $a+$ or later $a+1$. I have also changed the wording of postulate 5. Stolz and Gmeiner use "Zahl," 1 and $a+1$, for N, E and $a\circ$, and add another postulate, to the effect that every number shall be called equal to itself.

1. *The special object E shall belong to the class N.*
2. *If an object a belongs to N then its successor $a\circ$ shall also belong to N.*
3. *If an object a belongs to N then its successor $a\circ$ shall be different from E.*
4. *If $a\circ = b\circ$ then $a = b$.*
5. *If a is any given element of N, then a can be found in the sequence E, $E\circ$, $(E\circ)\circ$, . . .*

The meaning of these postulates will be clearer, if we consider at once the following systems, in which N, E and $\circ$ are so chosen that all the postulates, or all but one, are satisfied.

(a) N = class of positive integers; $E = 1$; $a\circ = a + 1$.
(b) N = class of positive integers with 0; $E = 0$; $a\circ = a + 1$.
(c) N = class of positive integral powers of 2;

$$E = 2^1;\ a\circ = 2a.$$

Each of these systems (a), (b), (c) satisfies all the five postulates.

(S_1) N = class of positive integers; $E = 0$; $a\circ = a + 1$.
(S_2) N = class of positive integers from 1 to 9;

$$E = 1;\ a\circ = a + 1.$$

(S_3) N = class of positive integers from 1 to 9;

$$E = 1;\ a\circ = a + 1, \text{ except that } 9\circ = 1.$$

(S_4) N = class consisting of two elements 0 and 1;

$$E = 0;\ 0\circ = 1,\ 1\circ = 1.$$

(S_5) N = class of positive integers; $E = 1$; $a\circ = a + 2$.

Here the system S_k ($k = 1, 2, 3, 4, 5$) satisfies all the other postulates, but not the kth.

The possibility of constructing such systems as (a), (b), (c) proves that *the postulates 1–5 are consistent;* and the consideration of the systems S_1–S_5 shows us that *the postulates 1–5 are independent of one another.* This proof of independence, by the way, which forms a most interesting feature of Peano's work, is not reproduced by Stolz and Gmeiner.

Further, it can be easily shown that any two systems, S and S', which satisfy all the postulates 1–5 can be brought into one-to-one correspondence with each other in such a way that

E will correspond with E', and whenever a in S corresponds with a' in S' then $a\circ$ will correspond with $a'\circ$. This fact may be stated in the theorem: *All systems* $(N, E, \circ)$ *which satisfy postulates 1–5 are equivalent* to one another (cf. the systems *a–c* above). On the basis of this theorem Peano then states his definition:

A "system of natural numbers" shall mean any system $(N, E, \circ)$ *which satisfies the postulates 1–5.*

It is clear that such a definition is a very different thing from a psychological analysis of the process of counting. It is rather the construction of a symbolic "calculus of operations," which contains indeed the arithmetic calculus as a special case when a special interpretation, viz., (a), is given to its symbols, but in which the validity of the deductions is quite independent of any such interpretation.

The first step in the development of the theory is the definition of the symbol $a + b$ as follows: Let a and b be any two natural numbers, that is, any two elements of a system $(N, E, \circ)$ which satisfies Peano's five postulates; then $a + b$ denotes the number determined by the following recursion formulæ:

$$a + E = a\circ; \quad a + (E\circ) = (a + E)\circ; \quad \cdots; \qquad a + (k\circ) = (a + k)\circ. \tag{6}$$

Thus, in order to compute the value of $a + b$ we must compute in succession the values of

$$a + E, \quad a + (E\circ), \quad a + [(E\circ)\circ], \quad \cdots, \tag{6'}$$

until we strike the desired element $a + b$.

The statement of Stolz and Gmeiner, defining $a + b$ as that number which is obtained by starting from a and proceeding b steps forward in the series of numbers, would seem to imply that the *ability to count* is after all involved in this process. Such is, however, not the case. In order to find the number $a + b$ by Peano's definition we require simply the ability to determine from any given element a the element $a\circ$; we can thus compute one after the other the elements of the sequence (6'), and we know by postulate 5 that this sequence will contain the element $a + b$. How far we shall have to go to find this element we do not know; we do not count the steps already taken at any stage, nor do we ask how many more there are to

take; it is sufficient to know that if we persevere, one step at a time, the desired number will surely be reached.*

The element $a + b$ thus determined is called the "sum" of the given elements a and b; when the system bears the interpretation (a) then $a + b$ will clearly be the sum of a and b in the ordinary sense.

Similarly, the "product," ab, of two elements a and b is defined by the recursion formulæ

$$aE = a; \quad a(E\circ) = (aE) + a; \quad \cdots; \quad a(k\circ) = (ak) + a. \quad (7)$$

From the definition of $+$ follows next the theorem

$$a + (b + c) = (a + b) + c$$

the proof of which we give in full as a typical illustration of the use of postulate 5 (method of mathematical induction). Thus: the theorem is obviously true when $c = E$, by (6). And if it is true when $c\circ$ is any particular element k then it will be true also when $c = k\circ$.† Therefore, being true for $e = E$, it is true for $c = E\circ$, and hence for $c = (E\circ)\circ$, etc. But by 5 any given element can be reached in this way; hence the theorem is true for all values of c.

In a similar way the theorems

$$a + b = b + a, \quad (ab)c = a(bc), \quad ab = ba,$$

and

$$(a + b)c = ac + bc$$

are established.

The definition of the symbols $>$ and $<$ depends on the following theorem, which is proved also by the use of (6): If $a \neq b$ then there is either a number λ such that $a = b + \lambda$ or else a number μ such that $b = a + \mu$ (and not both). In the first case a is called "greater than" b $(a > b)$, in the second case "less" $(a < b)$. (8)

When the system bears the interpretation (a) the use of the symbols $>$ and $<$ as thus defined will clearly agree with the ordinary use of them.

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* Of course this implies unlimited time at our disposal—an assumption not explicitly stated by Peano or by Stolz and Gmeiner.

† For on the given assumption we have, by 6,

$$(a + b) + (k\circ) = [(a + b) + k]\circ = [a + (b + k)]\circ$$
$$= a + [(b + k)\circ] = a + [b + (k\circ)].$$

From the definition of $>$ and $<$ we have at once that $a + b > a$, and also the theorems: If $a > a'$ then $a + b > a' + b$ and $ab > a'b$. For example, from $a > a'$ follows $a = a' + x$, whence $ab = (a' + x)b = a'b + xb$, or $ab > a'b$. This proof of Peano's is simpler than the induction proof used by Stolz and Gmeiner.

After these fundamental definitions and theorems have been established the further development of the theory — including the introduction of the rational, irrational, negative and complex numbers — proceeds along lines already familiar and need not be enlarged upon here.

In a sense, the five postulates of Peano may be said to contain, implicitly, all the results of algebra and analysis, including the theory of functions of a complex variable. It should be clearly understood, however, that the business of the mathematical theory is solely to draw logical deductions from these postulates, not to discuss the question whether the results possess any real significance in the objective world. On this deeper question, Peano's work merely reduces the problem to its lowest terms; if we can show that his fundamental postulates are capable of real interpretation in the objective world, then all their consequences will likewise be capable of such interpretation. Whether, however, a real interpretation of the postulates—(for example, the system (*a*) on p. 42)—is psychologically possible or not, is a problem of epistemology with which the mathematical theory has nothing whatever to do.

A precisely similar remark applies to any deductive theory such as that contained in Hilbert's Grundlagen der Geometrie (1899). At the basis of Hilbert's theory stand certain symbols such as "point," "line," "between," etc., whose meaning is left wholly undetermined except for the imposition of certain postulates. The development of the theory consists in deducing the theorems which follow from the fundamental postulates by the laws of formal logic, quite independently of any special interpretation of the undefined symbols. To be sure, when the undefined symbols are interpreted as meaning point, line between, etc., in the ordinary sense then the whole theory becomes identical with our ordinary geometry of space. But the question whether such an interpretation is possible or not is a question which the deductive theory leaves wholly untouched. Hence Hilbert's theory cannot be said to give us any new knowledge of the real nature of space, any more than

Peano's theory gives us any new knowledge of the real nature of the natural numbers. Both these theories, however, enable us to state the philosophical problem with a definiteness which has not heretofore been possible.

Further comment on Stolz and Gmeiner's book is impossible in the space at our disposal. The remaining instalments of this great work will be awaited with keen interest.

E. V. HUNTINGTON.

HARVARD UNIVERSITY,
June 12, 1902.

LAZARUS FUCHS.*

FUCHS is dead. This announcement must have caused deep sorrow in the heart of many American mathematicians. For many of us have been his pupils, and to some of us his example has been the greatest inspiration of our lives. The writer of this little sketch is one of these. He remembers how he looked forward to the time when he would be fitted to attend Fuchs's lectures. He remembers the small and crowded lecture-room in the University of Berlin, poorly ventilated, stuffy and hot in the summer days, but so full of meaning and inspiration to the earnest and thoughtful student. Fuchs was not a brilliant lecturer. He spoke in a quiet, undemonstrative manner, but what he said was full of substance. To the student there was the inspiration of seeing a mathematical mind of the highest order full at work. For Fuchs worked when he lectured. He was rarely well prepared, but produced on the spot what he wished to say. Occasionally he would get lost in a complicated computation. Then he would look around at the audience over his glasses with a most winning and child-like smile. He was always certain of the essential points of his argument, but numerical examples gave him a great deal of trouble. He was fully conscious of this failing, and I remember well one occasion when, after a lengthy discussion, he laid considerable emphasis upon the fact that "*in this case*, two times two is four."

The mathematical papers of Fuchs are very numerous, but excepting a few of his earliest attempts, they are all connected directly or indirectly with the theory of linear differential equations. This was the province which, to quote the words of Auwers when he introduced Fuchs to the Berlin Academy of

* Immanuel Lazarus Fuchs, born in Moschin, near Posen, May 5, 1833, died at Berlin, April 26, 1902.

Sciences, he had added to the mathematical kingdom. Of course the conquest of this new territory had been prepared by others. The theory of functions, which was an essential prerequisite had been built up by Cauchy and Riemann. The work of Briot and Bouquet on differential equations of the first order was an illustration of the application of this new theory to the treatment of differential equations. But to Fuchs belongs the credit of first applying the theory of functions to the linear differential equations of any order, with rational coefficients. His paper in the sixty-sixth volume of *Crelle's Journal* is a classic, and to this day I know of no clearer exposition of the fundamental principles involved. It is true that Riemann was acquainted with these principles, as his posthumous paper on this subject proves. It is true also that Fuchs took his immediate inspiration from Riemann's famous paper on the hypergeometric series. But all of this does not lessen the credit due to Fuchs. To generalize is one of the functions of the mathematician. This Fuchs did. Riemann also did this, but his paper was never published until the theory had long been developed by Fuchs. It is interesting to notice in this connection the difference between the points of view adopted by Riemann and Fuchs. Fuchs starts out with the linear differential equation of the nth order whose coefficients are rational functions of x. By a rigorous examination of the convergence of the series, which formally satisfies the differential equation he finds that these equations have a very important and characteristic property. The singular points of the solutions are fixed, *i. e.*, they are independent of the constants of integration and can be found without first integrating the differential equation. They are in fact included among the poles of its coefficients. He then shows that a fundamental system of solutions undergoes a linear substitution with constant coefficients when the variable x describes a circuit enclosing such a singular point, and from this behavior of the solutions derives expressions for them, valid in a circular region surrounding that point and reaching as far as the next singular point. He thus establishes the existence of systems of n functions, uniform, finite and continuous, except in the vicinity of certain points, and undergoing linear substitutions with constant coefficients when the variable x describes circuits around these points.

Riemann's point of view was exactly opposite to this. With him it was a matter of principle not to define a function by an

expression, but by a characteristic property. He, therefore, starts out with the assumption of a system of n functions uniform, finite and continuous, except in the vicinity of certain arbitrarily assigned points, and undergoing an arbitrarily assigned linear substitution when the variable x describes a closed path around such a point. He then shows that such a system of functions will satisfy a linear differential equation of the nth order. But the theorem that the branch points and the substitutions may be arbitrarily assigned, ought to be proved at the outset if this point of view is to be adopted. Much has since been done on this question, which is really a fundamental one in the theory of linear differential equations.

This is not the place to go into details. Suffice it to say that the theory of linear differential equations was placed by Fuchs upon a solid and adequate foundation. He and his followers have reared upon this a noble structure. He himself characterized a class of such equations, called after him Fuchsian, whose solutions are everywhere regular. He studied the question of algebraic integrability which has so many points of contact with other questions of importance. He studied by his own methods the periods of an elliptic integral as function of the modulus, for Legendre had shown that these verify a linear differential equation of the second order. By considering the modulus inversely as a function of the quotient of the periods, a uniform automorphic function, now known as the modular function, was obtained. The theory of modular functions, and more generally of automorphic functions owes much to Fuchs, as well as to Klein and to Poincaré, who as an indication of this fact has named a large class of such functions Fuchsian. Fuchs has introduced other transcendental functions into analysis, connected with a linear differential equation in much the same way as the abelian functions are connected with an algebraic equation. Little has yet been done with these beyond the proof of their existence.

We have already mentioned that one of the first results obtained by Fuchs in his classical memoir was the fact that the branch points of a linear differential equation are fixed. Fuchs himself was the first to ask the question; are there other equations of this kind? In a beautiful paper in the *Sitzungsberichte der Berliner Akademie*, he started the investigation of this important question, confining himself, however, to differential equations of the first order. Poincaré completed the investiga-

tion in a remarkable manner, the result being that no essentially new functions could be defined by differential equations of the first order with fixed singular points. Painlevé has since then found that transcendental functions essentially new can be defined by such equations of order higher than the first.

We will close this brief sketch by translating a sentence, which is as characteristic of the modern theory of differential equations, as the famous definition of Kirchhoff is of modern mechanics. Fuchs says in his famous paper in *Crelle's Journal*: "In the present condition of science it is not so much the problem of the theory of differential equations to reduce a given differential equation to quadratures, as to deduce from the equation itself the behavior of its integrals at all points of the plane, *i. e.*, for all values of the complex variable."

This is the present point of view in the theory of differential equations. The first chapter of this theory, that of linear differential equations, has been far advanced, although not completed, by Fuchs and his pupils. Something has been done on later chapters, but not much. The theory of non-linear differential equations is one of the central problems of modern mathematics, but it has not yet found its Fuchs.

E. J. WILCZYNSKI.

UNIVERSITY OF CALIFORNIA,
BERKELEY, *July* 8, 1902.

NOTES.

THE fifty-first annual meeting of the American association for the advancement of science, was held at Pittsburgh, June 28–July 3. The total number of members in attendance was 431, and 320 papers and addresses were presented, of which 24 are classified under mathematics and astronomy (Section A). Among the officers elected for the coming year are: President of the Association, Dr. IRA REMSEN; Vice-president of Section A, Professor G. B. HALSTED; Secretary of Section A, Professor C. S. HOWE. A report of the meeting of Section A will appear in an early number of the BULLETIN. It was decided to hold the next meeting of the Association at Washington, D. C., in convocation week, December 29, 1902 – January 3, 1903.

IN its annual list of doctorates conferred by American universities during the academic year closing with June, 1902, *Science*

records the following with theses in mathematics: A. B. COBLE, Johns Hopkins University, "A study of the ternary quartic in its relation to the conics"; OTTO DUNKEL, Harvard University, "Regular singular points of a system of homogeneous linear differential equations of the first order"; PETER FIELD, Cornell University, "The forms of unicursal quintic curves"; CARL GUNDERSEN, Columbia University, "On the measure or content of assemblages of points"; C. J. KEYSER, Columbia University, "The plane geometry of the point in space of four dimensions"; T. M. PUTNAM, University of Chicago, "Concerning the linear fractional group on three variables with coefficients in the Galois field of order p^n"; H. L. RIETZ, Cornell University, "On primitive groups of odd order"; W. E. TAYLOR, Syracuse University, "On the product of an alternant by a symmetric function."

OXFORD UNIVERSITY. — The following courses in mathematics are announced for Michaelmas term, 1902: — By Professor E. B. ELLIOTT: Theory of numbers, two hours; Theory of functions, one hour. — By Professor H. H. TURNER: Elementary mathematical astronomy, two hours. — By Professor W. ESSON: Analytic geometry of plane curves, two hours; Synthetic geometry of plane curves, one hour. — By Professor A. E. H. LOVE: Spherical harmonics and other methods of analysis that are appropriate in applications to physics, three hours. — By Mr. C. E. HASELFOOT: Algebra, two hours. — By Mr. C. LEUDESDORF: Projective geometry, three hours. — By Mr. A. E. JOLLIFFE: Analytical geometry, two hours. — By Mr. J. W. RUSSELL: Differential calculus, two hours. — By Mr. R. F. MCNEILE: Curve tracing, one hour. — By Mr. P. J. KIRKBY: Introduction to higher algebra, one hour. — By Mr. A. L. PEDDER: Problems in pure mathematics, one hour. — By Mr. C. H. SAMPSON: Solid geometry, two hours. — By Mr. J. E. CAMPBELL: Differential equations, two hours. — By Mr. C. H. THOMPSON: Integral calculus, two hours. — By Mr. E. H. HAYES: Analytical statics, three hours. — By Mr. A. L. DIXON: Hydrostatics, one hour. — By Mr. H. T. GERRANS: Advanced rigid dynamics, two hours.

Considerable changes will be made in the First Public Examination for those who seek honors in mathematics by a scheme which comes into effect in 1904. A new subject, viz:

The elements of analytical geometry of three dimensions, is to be introduced; and restrictions on the freedom of choice of methods are to be removed. Under the existing scheme the elements of the mechanics of solids and fluids is treated without the aid of the infinitesimal calculus; in future the use of the calculus in its applications to mechanics is to be encouraged. At present there is a separation of geometry into two papers, devoted respectively to analytical and synthetic methods; in future candidates are to be encouraged to choose the method, analytical or synthetic, which appears the more appropriate in answering any question. The examination is taken in the first year or the second year of the academic course, and successful candidates have the option of proceeding to a final honors school in mathematics, or in physics, or in any other subject; in these respects the new scheme makes no changes.

THE several German universities below offer during the winter semester, 1902–1903, courses in mathematics as follows:

UNIVERSITY OF BERLIN. — By Professor H. A. SCHWARZ: Differential calculus, four hours, with exercises, two hours; Maxima and minima in elementary geometrical treatment, two hours; Applications of the theory of elliptic functions, four hours; Colloquium, two hours; Seminar, three hours. — By Professor G. FROBENIUS: Theory of determinants, four hours; Algebra, four hours; Seminar, three hours. — By Professor R. LEHMANN-FILHÉS: Analytical geometry, four hours; Theory of probabilities, one hour. — By Professor J. KNOBLAUCH: Theory of space curves and curved surfaces, four hours; Analytical mechanics, four hours; Proseminar, one hour. — By Professor K. HENSEL: Integral calculus, four hours, with exercises, two hours; Theory of elliptic functions, four hours. — Professor G. HETTNER: Theory of definite integrals, two hours. — By Dr. E. LANDAU: Theory of numbers, four hours; Theory of linear differential equations, four hours; Mengenlehre, one hour.

UNIVERSITY OF BONN. — By Professor L. HEFFTER: Analytical mechanics, four hours; Plane analytical geometry, four hours, with exercises, one hour. — By Professor H. KORTUM: Elliptic functions, four hours; Seminar, two hours. — By Professor R. LIPSCHITZ: Integral calculus II, four hours; Seminar, two hours. — By Professor J. SOMMER: Algebra, two hours; Geometrical applications of the theory of functions,

two hours. — By Dr. WENTSCHER: Philosophy of mathematics, one hour.

UNIVERSITY OF BRESLAU. — By Professor J. ROSANES: Differential calculus and elements of integral calculus, four hours; Theory of invariants, two hours; Seminar, one hour. — By Professor R. STURM: History of mathematics, one hour; Seminar, two hours; Theory of numbers, three hours; Geometrical loci of higher degree, three hours. — By Professor F. LONDON: Introduction to the theory of differential equations, three hours: Mathematics of insurance, two hours.

UNIVERSITY OF ERLANGEN. — By Professor P. GORDAN: Analytical geometry, four hours; Invariants, four hours; Seminar, three hours. — By Professor M. NÖTHER: Infinitesimal calculus, four hours; Theory of functions, four hours. — Seminar.

UNIVERSITY OF FREIBURG. — By Professor J. LÜROTH: Analytical mechanics, four hours; Least squares, two hours; Seminar. — By Professor L. STICKELBERGER: Plane analytical geometry and differential calculus, five hours, with exercises, two hours. — Theory of analytic functions, three hours. — By Professor A. LOEWY: Theory of curves and surfaces, four hours; Mathematics of insurance, two hours. — By Dr. E. REBMANN: History of arithmetic, two hours.

UNIVERSITY OF GÖTTINGEN. — By Professor F. KLEIN: Encyclopædia of mathematics, four hours; Seminar (with Professor Bohlmann), two hours. — By Professor D. HILBERT: Differential and integral calculus II, four hours; Mechanics of continua, four hours; Seminar in the theory of functions, two hours. — By Professor K. SCHWARZSCHILD: Celestial mechanics, three hours, with exercises, two hours. — By Professor H. MINKOWSKI: Algebra, three hours; Analysis situs, two hours; Seminar in the theory of functions, two hours. — By Professor M. BRENDEL: Introduction to theoretical astronomy, three hours; Life and work of Gauss, one hour. — By Professor F. SCHILLING; Analytical theory of curved lines and surfaces, four hours, with exercises, two hours. — By Professor G. BOHLMANN: Theory of probabilities, two hours; Theory of risks, two hours; Seminar, two hours; Seminar in the mathematics of insurance, two hours. — By Dr. E. ZERMELO: Theory of functions, four hours; Exercises in integral calculus and potential theory, two hours. — By Dr. BLUMENTHAL: Abelian func-

tions, two hours; Introduction to higher mathematics for students of science, with exercises, three hours.

UNIVERSITY OF HEIDELBERG. — By Professor L. KOENIGSBERGER: Higher algebra, theory of algebraic equations, four hours; Elements of the theory of differential equations, two hours; Calculus of variations, one hour; Theory of numbers, one hour; Seminar, two hours. — By Professor H. VALENTINER: Theory of orbits, three hours. — By Professor M. CANTOR: Differential and integral calculus, four hours, with exercises, one hour; Political arithmetic, two hours. — By Professor F. EISENLOHR; Theoretical optics, four hours; Differential and integral calculus, five hours; Potential, two hours. — By Professor K. KOEHLER: Synthetic geometry of space, three hours. — By Professor G. LANDSBERG: Descriptive geometry, with exercises, four hours; Theory of functions, three hours. — By Dr. K. BOEHM: Theory of partial differential equations, one hour; Vector analysis, one hour; D'Alembert's memoir on dynamics, one hour.

UNIVERSITY OF JENA. — By Professor J. THOMAE: Application of infinitesimal calculus to geometry, four hours; Definite integrals and Fourier series, four hours; Seminar, two hours. — By Professor A. GUTZMER: Analytical geometry of space, four hours; Analytical mechanics, four hours, with seminar, one hour. — By Professor J. FREGE: Differential and integral calculus II, with exercises, five hours.

UNIVERSITY OF KIEL. — By Professor L. POCHHAMMER: Elements of the theory of numbers, three hours; Introduction to the theory of functions, three hours; Seminar one hour. — By Professor P. HARZER: Celestial mechanics, three hours. — By Professor P. STÄCKEL: Integral calculus, three hours; Differential geometry of curved surfaces, three hours; Intrinsic geometry of curves, one hour; Seminar, one hour.

UNIVERSITY OF LEIPSIC. — Professor SCHEIBNER offers no lectures during the winter semester. — By Professor C. NEUMANN: Differential and integral calculus, four hours; Seminar, two hours. — By Professor H. BRUNS: Theory of probabilities, four hours; Celestial mechanics, two hours. — By Professor A. MAYER: Calculus of variations, four hours. — By Professor O. HÖLDER: Elliptic functions, four hours; Partial differential equations, two hours; Seminar, one hour. — By Professor F.

ENGEL: Determinants and algebraic equations, four hours; Theory of transformation groups (continuation), two hours; Seminar, two hours. — By Dr. F. HAUSDORFF: Analytical mechanics, three hours, with exercises, one hour. — By Dr. H. LIEBMANN: Analytical geometry of space, two hours; Theory of definite integrals, two hours; Descriptive geometry, two hours, with exercises, one hour.

UNIVERSITY OF STRASSBURG. — By Professor T. REYE: Geometry of position, three hours; Analytical mechanics, two hours; Seminar, two hours. — By Professor H. WEBER: Partial differential equations of mathematical physics, four hours; Selected chapters of algebra, two hours; Seminar, two hours. By Professor F. ROTH: Algebraic analysis and determinants, three hours; Analytical geometry of space, two hours; Ordinary differential equations, two hours. — By Professor A. KRAZER: Infinitesimal calculus, four hours; Analytical geometry of the plane, three hours; Seminar, two hours.

THE London mathematical society awarded its De Morgan medal for 1902 to Professor A. G. GREENHILL.

PROFESSORS ORMOND STONE, of the University of Virginia, E. H. MOORE, of the University of Chicago, and FRANK MORLEY, of Johns Hopkins University, have been appointed by the executive committee of the Carnegie Institution, as advisors in relation to original research in mathematics.

PROFESSOR W. E. STORY, while retaining his chair in Clark University, has been appointed head of the mathematical department in the newly established Clark College, Worcester, Mass. Mr. F. H. HODGE, at present fellow in mathematics in Clark University, becomes instructor in mathematics in the college.

AT the University of Nebraska, Dr. R. E. MORITZ, who recently returned from a year's study abroad, has been appointed to an adjunct professorship of mathematics.

DR. CHARLES W. M. BLACK, instructor in mathematics in the University of Oregon, died August 11, at La Grande, Ore.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BELTRAMI (E.). Opere matematiche, pubblicate per cura della Facoltà di scienze della r. Università di Roma. Vol. I. Milano, Hoepli, 1902. 4to. 22 + 438 pp. With portrait. Fr. 25.00

BERTRAND (J.). Eloges académiques. Nouvelle série: Poinsot, Cosson, Chasles, Cordier, Paris, Cauchy, Tisserand, Viète, Galilée, D. Papin, Clairaut, Euler, d'Alembert et Lagrange, Abel, Galois, Faraday, Pasteur. Avec un éloge historique de Joseph Bertrand, par G. Darboux. Paris, Hachette, 1902. 16mo. 51 + 411 pp. Fr. 3.50

BIANCHI (L.). Lezioni di geometria differenziale. 2a edizione, riveduta e considerevolmente aumentata, in due volumi. Vol. I. Pisa, Spoerri, 1902. 8vo. 524 pp. Fr. 18.00

BOLZA (O.). Concerning the geodesic curvature and the isoperimetric problem on a given surface and proof of the sufficiency of Jacobi's condition for a permanent sign of the second variation in the so-called isoperimetric problems. Printed from Vol. 9 of "The decennial publications." Chicago, University of Chicago Press, 1902. 4to. 2 + 7 pp. (University of Chicago decennial publications.) $0.25

BONNESEN (T.). Analytiske studier over ikke-Euklidisk geometri. Kjöbenhavn, Ursius, 1902. 8vo. 103 pp. M. 3.60

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